

STABLE \mathbb{A}^1 -CONNECTIVITY OVER DEDEKIND SCHEMES

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ABSTRACT. We show that \mathbb{A}^1 -localization decreases the Nisnevich-stalkwise connectivity by at most one over a Dedekind scheme with infinite residue fields. For the proof, we establish a Nisnevich-local version of Gabber's geometric presentation lemma over a henselian discrete valuation ring with infinite residue field.

INTRODUCTION

Background. In [Mor05], Morel proved his celebrated *stable \mathbb{A}^1 -connectivity theorem*: The \mathbb{A}^1 -localization of a Nisnevich-stalkwise connected spectrum on the smooth Nisnevich-site over a field is still Nisnevich-stalkwise connected. For more general regular base schemes, Morel calls the analogous property the *stable \mathbb{A}^1 -connectivity property*. In [Mor05, Conj. 2], he conjectured that this property holds over every regular base. However, in [Ayo06], Ayoub constructed a counterexample to this conjecture (see Remark 4.7 below).

Aim and results. In this paper, we want to replace the stable \mathbb{A}^1 -connectivity property by the following weaker property: A base scheme S of Krull-dimension d has the *shifted stable \mathbb{A}^1 -connectivity property*, if \mathbb{A}^1 -localization of a Nisnevich-stalkwise d -connected spectrum is at least still Nisnevich-stalkwise connected. In other words, S has this property, if \mathbb{A}^1 -localization lowers the stalkwise connectivity by at most the dimension of S . Question 4.8 below asks, if every regular base scheme has this shifted stable \mathbb{A}^1 -connectivity property. Morel's connectivity theorem [Mor05, Theorem 6.1.8] is a positive answer in the case $d = 0$. In the main theorem of this paper, we give a positive answer in the one-dimensional case, assuming infinite residue fields (cf. Corollary 4.15):

Theorem 1. *A Dedekind scheme with only infinite residue fields has the shifted stable \mathbb{A}^1 -connectivity property: If E is a Nisnevich-stalkwise i -connected spectrum, then its \mathbb{A}^1 -localization $L^{\mathbb{A}^1}E$ is stalkwise $(i - 1)$ -connected.*

Examples for such base schemes are algebraic curves over infinite fields in geometric settings or $\mathrm{Spec}(\mathbb{Z}_p^{\mathrm{nr}})$ for $\mathbb{Z}_p^{\mathrm{nr}}/\mathbb{Z}_p$ the maximal unramified extension in more arithmetic settings.

As in Morel's case, the proof of this theorem needs the following strong geometric input of independent interest. In Chapter 2, we prove a Nisnevich-local version of Gabber's geometric presentation result [CTHK97, Theorem 3.1.1] over a henselian discrete valuation ring with infinite residue fields (cf. Theorem 2.1):

Theorem 2. *Let \mathfrak{o} be a henselian discrete valuation ring with infinite residue field. Let X/\mathfrak{o} be a smooth \mathfrak{o} -scheme of finite type and let $Z \hookrightarrow X$ be a proper closed subscheme. Let z be a point in Z . If z lies in the special fibre, suppose that $Z_\sigma \neq X_\sigma$. Then, Nisnevich-locally around z , there exists a smooth \mathfrak{o} -scheme V of finite type and a Nisnevich-distinguished square*

$$\begin{array}{ccc} X \setminus Z & \longrightarrow & X \\ \downarrow & & \downarrow p \\ \mathbb{A}_V^1 \setminus p(Z) & \longrightarrow & \mathbb{A}_V^1 \end{array}$$

such that Z is finite over V .

The proof is based on [CTHK97, Theorem 3.1.1] combined with a Noether normalization over a Dedekind base (cf. [Kai15, Theorem 4.6]).

Besides this geometric input, we need a second key ingredient of a more homotopical kind: In Chapter 3, we examine a vanishing result for the non-sheafified homotopy classes of the \mathbb{A}^1 -localization of a

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stalkwise connected spectrum. This is a slight generalization of the argument in [Mor05, Lem. 4.3.1] to arbitrary base schemes. As a byproduct, we obtain that the S^1 - and the \mathbb{P}^1 -homotopy t-structure over any base scheme is left-complete, i.e., a presheaf of spectra is recovered as the homotopy limit over its Postnikov truncations (see Corollary 3.6 and 3.8).

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1. PRELIMINARIES

In this paper, our base scheme S is always a noetherian scheme of finite Krull-dimension. Let Sm_S be the category of smooth schemes of finite type over S . The category Sm_S is essentially small and sometimes we choose a small skeleton implicitly without mentioning. Let $\mathrm{sPre}_+(S)$ be the category of pointed simplicial presheaves on Sm_S . We will mostly ignore S in the notation. For an object $U \in \mathrm{Sm}_S$ let U_+ denote be presheaf $\mathrm{hom}_{\mathrm{Sm}_S}(-, U)$ considered as a discrete simplicial set with an additional disjoint basepoint. Whenever we speak of a category having all limits and colimits we actually mean that it has all *small* limits and all *small* colimits.

Model structures. In contrast to the foundational address [MV99] of \mathbb{A}^1 -homotopy theory, we use projective analogues of the unstable model structures and obtain the (pointed) *objectwise*, *Nisnevich-local* and \mathbb{A}^1 -*Nisnevich-local* model structure. Throughout the whole text, let L^{ob} , L^s and $L^{\mathbb{A}^1}$ denote fixed (pointed) objectwise, Nisnevich-local and \mathbb{A}^1 -Nisnevich-local fibrant replacement functors, respectively. For a symbol $\tau \in \{ob, s, \mathbb{A}^1\}$, a non-negative integer n and $F \in \mathrm{sPre}_+$, define the n -th τ -homotopy presheaf of F as

$$\pi_n^\tau(F)(-) := [(-)_+ \wedge S^n, L^\tau F]$$

and the n -th τ -homotopy sheaf $\tilde{\pi}_n^\tau(F)$ of F as its Nisnevich-sheafification. Here, the brackets denote (pointed) objectwise homotopy classes. Note that $\tilde{\pi}_n^{ob}(F) \cong \tilde{\pi}_n^s(F)$. Whenever the objectwise model structure is considered, we omit the symbol ob from the notation.

Let $\mathrm{Spt}_{S^1}(S)$ be the category of S^1 -spectra on the category $\mathrm{sPre}_+(S)$. The functor $(-)_0$ sending an S^1 -spectrum to its zeroth level and the S^1 -suspension spectrum functor $\Sigma_{S^1}^\infty$ fit into an adjunction $\Sigma_{S^1}^\infty : \mathrm{sPre}_+ \rightleftarrows \mathrm{Spt}_{S^1} : (-)_0$. For an integer $n \geq 0$, there is also an adjunction $[-n] : \mathrm{Spt}_{S^1} \rightleftarrows \mathrm{Spt}_{S^1} : [n]$ of shift functors with $E[n]_m := E_{n+m}$, where we define $E_m := *$ for $m < 0$. Following the general procedure of [Hov01], we equip the category $\mathrm{Spt}_{S^1}(S)$ with stable model structures having homotopy categories $\mathcal{SH}_{S^1}^{ob}(S)$, $\mathcal{SH}_{S^1}^s(S)$ and $\mathcal{SH}_{S^1}^{\mathbb{A}^1}(S)$, respectively. The two above-mentioned adjunctions turn into Quillen adjunctions. Each of these stable homotopy categories is a triangulated category with distinguished triangles given by the homotopy cofibre sequences [Hov99, Prop. 7.1.6]. In fact, by choosing symmetric spectra as a more elaborate model, these homotopy categories carry the structure of a closed symmetric monoidal category with a compatible triangulation in the sense of [May01, Def. 4.1]. For details of this construction we refer to [Hov01], [Jar00] and [Ayo07]. There are functors

$$- \wedge \Sigma_{S^1}^\infty(-) : \mathrm{Spt}_{S^1} \times \mathrm{sPre}_+ \rightarrow \mathrm{Spt}_{S^1}$$

$$\underline{\mathrm{hom}}(\Sigma_{S^1}^\infty(-), -) : \mathrm{sPre}_+^{op} \times \mathrm{Spt}_{S^1} \rightarrow \mathrm{Spt}_{S^1}$$

defined in the obvious way. For a cofibrant $F \in \mathrm{sPre}_+$, they fit into a Quillen adjunction

$$- \wedge \Sigma_{S^1}^\infty F : \mathrm{Spt}_{S^1} \rightleftarrows \mathrm{Spt}_{S^1} : \underline{\mathrm{hom}}(\Sigma_{S^1}^\infty F, -)$$

whose derived adjunction models the monoidal structure from before.

We mention that a concrete fibrant replacement functor for the stable τ -model structure on Spt_{S^1} is given by

$$\Theta_{S^1}^\tau E = \mathrm{colim}((L^\tau E) \rightarrow \Omega_{S^1}(L^\tau E)[1] \rightarrow (\Omega_{S^1})^2(L^\tau E)[2] \rightarrow \dots)$$

where $\tau \in \{ob, s, \mathbb{A}^1\}$ and where the application of L^τ to a spectrum is levelwise [Hov01, Theorem 4.12]. We write $\Omega_{S^1}^\infty : \mathrm{Spt}_{S^1} \rightarrow \mathrm{sPre}_+$ for the composition of this fibrant replacement functor with $(-)_0$.

As for the unstable structures, we define the n -th stable τ -homotopy presheaf of $E \in \mathrm{Spt}_{S^1}$ as

$$\pi_n^\tau(E)(-) := [\Sigma_{S^1}^\infty(-)_+[n], L^\tau E]$$

and the n -th stable τ -homotopy sheaf $\tilde{\pi}_n^\tau(E)$ of E as its Nisnevich-sheafification. Here, $[-, -]$ denotes the morphism sets of $\mathcal{SH}_{S^1}^{ob}$. Again, note that $\tilde{\pi}_n^{ob}(E) \cong \tilde{\pi}_n^s(E)$. Since it will be evident from the context if the unstable or the stable homotopy sheaf is considered, we do not introduce an extra decoration.

We use the following explicit model for $L^{\mathbb{A}^1}$ in the stable context introduced in [Mor04] and [Mor05].

Theorem 1.1 (Morel). *Let S be an arbitrary base scheme. For each integer $k \geq 0$, we set $L^k(E) := \underline{\mathrm{hom}}(F^{\wedge k}, L^s(E))$ with $F := \Sigma_{S^1}^\infty C[-1]$ where C is a cofibrant replacement of the cofibre of the morphism*

$$S^0 \xrightarrow{0,1} \mathbb{A}^1$$

in sPre_+ . Then the functor $L^\infty: \mathrm{Spt}_{S^1} \rightarrow \mathrm{Spt}_{S^1}$ defined by

$$L^\infty(E) := \mathrm{hocolim}_{k \rightarrow \infty} L^k(E)$$

is a fibrant replacement functor for the stable \mathbb{A}^1 -Nisnevich-local model.

Remark 1.2. Likewise, the spectrum F from the above Theorem 1.1 may be defined by the distinguished triangle

$$F \rightarrow \Sigma_{S^1}^\infty S^0 \xrightarrow{0,1} \Sigma_{S^1}^\infty \mathbb{A}^1.$$

Let $k \geq 1$ be an integer. After rotation and smashing with the spectrum $F^{\wedge(k-1)}$, the above triangle becomes $\Sigma_{S^1}^\infty \mathbb{A}^1 \wedge F^{\wedge(k-1)}[-1] \rightarrow F^{\wedge k} \rightarrow F^{\wedge(k-1)}$. Applying $\underline{\mathrm{hom}}(-, L^s(E))$ yields the distinguished triangle

$$L^{k-1}(E) \rightarrow L^k(E) \rightarrow \underline{\mathrm{hom}}(\Sigma_{S^1}^\infty \mathbb{A}^1, L^{k-1}(E)[1]).$$

Here, $L^{k-1}(E)[1] \simeq L^{k-1}(E[1])$ holds by definition and homotopy-exactness of L^s .

Basechange. We briefly recall the construction of base change functors in \mathbb{A}^1 -homotopy theory. For details consider the monograph [Ayo07] and [Hu01].

Let $f: R \rightarrow S$ be a morphism between noetherian schemes of finite Krull-dimension. There is an adjunction

$$f^*: \mathrm{sPre}_+(S) \rightleftarrows \mathrm{sPre}_+(R): f_*$$

where the *direct image* functor f_* is defined by $(f_*G)(-) := G(- \times_S R)$ and where the left-adjoint *inverse image* is determined by $f^*(U_+) := (U \times_S R)_+$ for $U \in \mathrm{Sm}_S$. Both functors are strong symmetric monoidal and there is a natural isomorphism $f_* \underline{\mathrm{hom}}_+(f^*F, G) \cong \underline{\mathrm{hom}}_+(F, f_*G)$ [FHM03, 3.4]. If the morphism $f: R \rightarrow S$ is in Sm_S , the inverse image functor has a left-adjoint

$$f_\sharp: \mathrm{sPre}_+(R) \rightleftarrows \mathrm{sPre}_+(S): f^*$$

determined by $f_\sharp(V_+ \rightarrow R) := V \sqcup S \rightarrow R \sqcup S \rightarrow S$. In this case, the inverse image is given by $f^*F = F \wedge_S R_+$, one has a *projection formula* $f_\sharp(G \wedge f^*F) \cong f_\sharp G \wedge F$ and a natural isomorphism $f^* \underline{\mathrm{hom}}_+(A, B) \cong \underline{\mathrm{hom}}_+(f^*A, f^*B)$ [FHM03, Prop. 4.9]. Note that, since S is noetherian, any open immersion $R \hookrightarrow S$ is in Sm_S .

The adjunction (f^*, f_*) is a Quillen adjunction for the objectwise, the Nisnevich-local and the \mathbb{A}^1 -Nisnevich-local model structures. If $f: R \rightarrow S$ is in Sm_S , then the adjunction (f_\sharp, f^*) is a Quillen adjunction for the objectwise, the Nisnevich-local and the \mathbb{A}^1 -Nisnevich-local model structures as well and f^* preserves all weak equivalences (cf. [Ayo07, Theoreme 4.5.10]).

Remark 1.3. For the projective versions of the model structures, it is easy to see that f^* and f_\sharp preserve the generating cofibrations and hence all cofibrations. By the same reason, the objectwise acyclic cofibrations are preserved, so (f_\sharp, f^*) and (f^*, f_*) are Quillen adjunctions for the objectwise structures. For to see that the right-adjoints f_* and f^* preserves fibrations for the Nisnevich-local and the \mathbb{A}^1 -Nisnevich-local model, it suffices to show that they preserve fibrations between fibrant objects [Dug01, Cor. A.2]. As the right-adjoint preserve objectwise fibrations, it suffices to show that they preserve fibrant objects. The fibrant objects of a Bousfield localization may be detected by a particular set J' of acyclic cofibrations [Hir03, Lem. 3.3.11]. It remains to be shown that the left-adjoints preserve these acyclic cofibrations in J' which is straightforward (cf. [DR03b, Def. 2.14]).

In particular, Remark 1.3 implies the following Lemma.

Lemma 1.4. *Let $f: R \rightarrow S$ be a morphism in Sm_S . For each $F \in \mathrm{sPre}_+(S)$, there are canonical (objectwise) weak equivalences*

$$L^s(f^*F) \sim f^*(L^s F) \quad \text{and} \quad L^{\mathbb{A}^1}(f^*F) \sim f^*(L^{\mathbb{A}^1} F)$$

in $\mathrm{sPre}_+(R)$.

The spectrum D of a henselian local ring of a point $s \in S$ is usually not an object of Sm_S itself. Hence, Lemma 1.4 does not apply directly to the canonical morphism $D \rightarrow S$. The following Lemma 1.5 fixes this problem.

Lemma 1.5. *Let $d: D \rightarrow S$ be the limit $D \in \text{Sch}_S$ of a cofiltered diagram $\underline{D}: \mathcal{I} \rightarrow \text{Sm}_S$ with affine transition morphisms, where $d_i: D_i \rightarrow S$ denotes the structure morphism of each $D_i := \underline{D}(i)$. Let $V \rightarrow D$ be an element of Sm_D (in particular, V/D is of finite presentation). Then, the following statements hold:*

- (1) *There is a final functor $\mathcal{I}_V \rightarrow \mathcal{I}$, a cofiltered diagram $\underline{V}: \mathcal{I}_V \rightarrow \text{Sm}_S$ with affine transition morphisms and a natural transformation $\underline{V} \rightarrow \underline{D}|_{\mathcal{I}_V}$ inducing $V \rightarrow D$ on the limit over \mathcal{I}_V in Sch_S .*
- (2) *For each $F \in \text{sPre}_+(S)$, the canonical morphism $\Gamma(\underline{V}, d^*F) \rightarrow \Gamma(V, d^*F)$ of diagrams induces a natural isomorphism*

$$\Gamma(V, d^*F) \cong \text{colim}_{i \in \mathcal{I}_V} \Gamma(V_i, d_i^*F).$$

- (3) *For each $F \in \text{sPre}_+(S)$, there is a canonical natural isomorphism of pointed (objectwise) homotopy classes*

$$[V_+, d^*F] \cong \text{colim}_{i \in \mathcal{I}_V} [V_{i+}, d_i^*F]$$

- (4) *In (1), open embeddings, étale morphisms, smooth morphisms resp. Nisnevich-distinguished squares in Sm_D can be approximated by sectionwise open embeddings, étale morphisms, smooth morphisms resp. Nisnevich-distinguished squares in Sm_S .*
- (5) *For each $F \in \text{sPre}_+(S)$, there are canonical (objectwise) weak equivalences*

$$L^s(d^*F) \sim d^*(L^sF) \quad \text{and} \quad L^{\mathbb{A}^1}(d^*F) \sim d^*(L^{\mathbb{A}^1}F)$$

in $\text{sPre}_+(D)$.

Proof. (1) Follows from [EGA4, Theorem 4.8.8.2, Prop. 4.17.7.8]. In fact, we may (and always will) even assume that $\mathcal{I}_V = \mathcal{I} \downarrow i_0$ for a suitable object $i_0 \in \mathcal{I}$ and $\underline{V} = V_{i_0} \times_{D_{i_0}} \underline{D}|_{\mathcal{I} \downarrow i_0}$ for a suitable smooth morphism $V_{i_0} \rightarrow D_{i_0}$.

For (2), we may assume F to be simplicially discrete, i.e., a presheaf. As we may write F as the colimit over representable presheaves and pullback- as well as section-functors preserve colimits in the category of presheaves, we may assume that F is representable by a suitable object $U \rightarrow S$ in Sm_S . Then $d^*F = U \times_S D$ resp. $d_i^*F = U \times_S D_i$ and (2) follows from (1) and [EGA4, Theorem 8.8.2].

For (3), let us first observe that d^* preserves objectwise fibrant objects. Indeed, this holds for the d_i^* by Remark 1.3. Taking sections and applying (2), it suffices to observe that a filtered colimit of fibrant simplicial sets is again fibrant. The assertion of (3) now follows by taking homotopies with respect to the functorial standard cylinder $(-) \times \Delta^1$.

For (4), let $f: V' \rightarrow V$ be an open embedding (resp. an étale or smooth morphism) in Sm_D . We apply (1) first to the structural map $V \rightarrow D$ of the target and then to $V' \rightarrow V$, itself. We get approximations $\underline{V}, \underline{V}': \mathcal{I}_f \rightarrow \text{Sm}_S$ and a natural transformation $\underline{f}: \underline{V}' \rightarrow \underline{V}$ inducing f after taking limits. By [EGA4, Prop. 8.6.3] (resp. [EGA4, Prop. 17.7.8]) we may assume that \underline{f} is sectionwise an open embedding (resp. an étale or smooth morphism).

If f is étale and $j: U \hookrightarrow V$ open inducing a Nisnevich-distinguished square, choose an approximation $\underline{j}: \underline{U} \hookrightarrow \underline{V}$ by sectionwise open embeddings in Sm_S starting with our original choice of \underline{V} . By [AM69, Prop. A.3.3] we may modify the index categories of \underline{f} and \underline{j} to get a uniform approximation

$$\begin{array}{ccc} \underline{U} \times_{\underline{V}} \underline{V}' & \longrightarrow & \underline{V}' \\ \downarrow & & \downarrow \underline{f} \\ \underline{U} & \xrightarrow{\underline{j}} & \underline{V} \end{array}$$

of the Nisnevich-distinguished square given by f and j with \underline{f} sectionwise étale and \underline{j} sectionwise an open embedding. In particular, the sectionwise definition of $\underline{f}^*(\underline{Z}) \rightarrow \underline{Z} := \underline{V} \setminus \underline{U}$ (sectionwise with the reduced structure) gives a well defined approximation of $f^*(Z) \rightarrow Z := V \setminus U$ (to see that that the diagram \underline{Z} is well defined, recall our assumptions in the first part of the proof). By [EGA4, Cor. 8.8.2.4] we may even assume that this approximation is sectionwise an isomorphism, i.e., the above square of approximations is sectionwise a Nisnevich-distinguished square.

For (5), note that both assertions are equivalent to d^* preserving Nisnevich-local resp. \mathbb{A}^1 -Nisnevich-local fibrant objects. Let $F \in \text{sPre}_+$ be Nisnevich-local fibrant. We have to show that d^*F sends Nisnevich-distinguished squares to homotopy pullback squares of simplicial sets. Let Q be a Nisnevich-distinguished square in Sm_D . By (4), Q may be approximated by a diagram \underline{Q} of Nisnevich-distinguished squares. By (2), we have $(d^*F)(Q) \cong \text{colim}(d_i^*F)(Q_i)$. Again, as the d_i^* admit Quillen left-adjoints for the Nisnevich-local model, it suffices to show that a filtered colimit of homotopy pullback squares of

simplicial sets is again a homotopy pullback square. This, in turn, follows from the fact that those colimits preserve categorical pullback squares, fibrations and weak equivalences of simplicial sets. By the previous observation for the Nisnevich-local structure, for the second assertion it suffices to show that d^* preserves \mathbb{A}^1 -invariant simplicial presheaves. This however, follows directly from (2) and the corresponding fact for the d_i^* since they admit $d_{i\sharp}$ as left-adjoints. \square

Corollary 1.6. *Let $F \in \mathbf{sPre}_+(S)$ be a simplicial presheaf. Then the following statements are equivalent:*

- (1) *The homotopy sheaf $\tilde{\pi}_0(F)$ is trivial.*
- (2) *For all schemes $V \in \mathbf{Sm}_S$ with structure morphism $p: V \rightarrow S$ and all points $v \in V$ with canonical morphism $\mathbf{v}: V_v^h := \mathrm{Spec}(\mathcal{O}_{V,v}^h) \rightarrow V$, the homotopy sheaf $\tilde{\pi}_0(\mathbf{v}^*p^*F)$ is trivial.*
- (3) *For all points $s \in S$ with canonical morphism $\mathbf{s}: S_s^h \rightarrow S$, the homotopy sheaf $\tilde{\pi}_0(\mathbf{s}^*F)$ is trivial.*

Proof. First suppose (2) holds. We want to show (1), i.e., we have to show that the Nisnevich-stalk at (V, v) of the sheaf $\tilde{\pi}_0(F)$ is trivial for all such (V, v) . By (3) of the previous Lemma 1.5, we get

$$\tilde{\pi}_0(F)_{(V,v)} = \operatorname{colim}_{f: (W,w) \rightarrow (V,v)} [W_+, f^*p^*F] \cong [V_{v+}^h, \mathbf{v}^*p^*F]$$

where the colimit runs over the Nisnevich-neighbourhoods of (V, v) . The identity $\mathrm{id}_{(V_v^h, v)}$ is cofinal in the Nisnevich-neighbourhoods of (V_v^h, v) , so we obtain $[V_{v+}^h, \mathbf{v}^*p^*F] = \tilde{\pi}_0(\mathbf{v}^*p^*F)_{(V_v^h, v)}$ which is trivial by assumption.

For implication (1) \Rightarrow (2), suppose that $\tilde{\pi}_0(F) = 0$. By Lemma 1.5.(4), every object in $\mathbf{Sm}_{V_v^h}$ has the form $W_v^h := W \times_{V_v^h} V$ for suitable $W \in \mathbf{Sm}_V$. Take a point $w \in W_v^h$. By abuse of notation, we denote its image in W by w as well. We observe that $(W_v^h)_w^h \cong W_w^h$. By Lemma 1.5, $\tilde{\pi}_0(\mathbf{v}^*p^*F)_{(W_v^h, w)} \cong \tilde{\pi}_0(F)_{(W, w)}$. Therefore, $\tilde{\pi}_0(\mathbf{v}^*p^*F)$ is trivial. As a special case we get implication (3) \Rightarrow (2). Finally, the reverse implication (2) \Rightarrow (3) is trivial. \square

For a morphism $f: R \rightarrow S$ of base schemes. There is also an adjunction

$$f^*: \mathbf{Spt}_{S^1}(S) \rightleftarrows \mathbf{Spt}_{S^1}(R): f_*$$

on the level of spectra where one defines $f^*(E)_n := f^*(E_n)$ and $f_*(D)_n := f_*(D_n)$ with obvious structure maps. If the morphism $f: R \rightarrow S$ is in \mathbf{Sm}_S , there is an adjunction

$$f_{\sharp}: \mathbf{Spt}_{S^1}(R) \rightleftarrows \mathbf{Spt}_{S^1}(S): f^*$$

with $f_{\sharp}(D)_n := f_{\sharp}(D_n)$. The adjunction (f^*, f_*) is a Quillen adjunction for all considered stable model structures. If $f: R \rightarrow S$ is in \mathbf{Sm}_S , then the adjunction (f_{\sharp}, f^*) is a Quillen adjunction for all considered stable model structures as well and f^* preserves all stable weak equivalences (cf. [Ayo07, Theorem 4.5.23]).

\mathbb{P}^1 -spectra. In this section, we will briefly recall a model for the \mathbb{P}^1 -stable *motivic homotopy category*. As an underlying category of this model structure, we use (\mathbb{G}_m, S^1) -bispectra, i.e., the category $\mathbf{Spt}_{\mathbb{G}_m}(\mathbf{Spt}_{S^1}(S))$ of \mathbb{G}_m -spectra with entries in \mathbf{Spt}_{S^1} . Here, by abuse of notation, \mathbb{G}_m denotes the S^1 -suspension spectrum of a cofibrant replacement of the pointed object $(\mathbb{G}_m, 1)$ of $\mathbf{sPre}_+(S)$ (cf. [Mor04, Rem. 5.1.10]). Again, by an abuse of notation, we abbreviate this category by $\mathbf{Spt}_{\mathbb{P}^1}(S)$ and call its objects \mathbb{P}^1 -spectra. Similarly to the passage from \mathbf{sPre}_+ to S^1 -spectra, the zeroth entry of an \mathbb{P}^1 -spectrum and the \mathbb{G}_m -suspension spectrum functor fit into an adjunction

$$(1.1) \quad \Sigma_{\mathbb{G}_m}^\infty: \mathbf{Spt}_{S^1} \rightleftarrows \mathbf{Spt}_{\mathbb{P}^1}: (-)_0.$$

For $q \geq 0$, there is also an adjunction $\langle -q \rangle: \mathbf{Spt}_{\mathbb{P}^1} \rightleftarrows \mathbf{Spt}_{\mathbb{P}^1}: \langle q \rangle$ of shift functors with $E\langle q \rangle_m = E_{q+m}$, where we define $E_m = *$ for $m < 0$. Again by the general procedure of [Hov01], we equip $\mathbf{Spt}_{\mathbb{P}^1}$ with the stable model structure induced via (1.1) by the stable \mathbb{A}^1 -Nisnevich-local structure on \mathbf{Spt}_{S^1} . Its homotopy category $\mathcal{SH}(S)$ is the $(\mathbb{P}^1$ -stable) *motivic homotopy category*. The two above-mentioned adjunctions turn into Quillen adjunctions for this structure.

The motivic homotopy category \mathcal{SH} is a triangulated category with distinguished triangles again given by the homotopy cofibre sequences. Note that here, the triangulated shift is again induced by the simplicial-shift $[1]$ and not by the \mathbb{G}_m -shift $\langle 1 \rangle$.

Finally, let us mention a concrete fibrant replacement functor for the above model structure on $\mathbf{Spt}_{\mathbb{P}^1}$. This is completely analogous to the S^1 -stabilization process from \mathbf{sPre}_+ to \mathbf{Spt}_{S^1} . Let $E \in \mathbf{Spt}_{\mathbb{P}^1}$. By [Hov01, Theorem 4.12], we may use the functor

$$\Theta_{\mathbb{G}_m} E = \operatorname{colim}(E \rightarrow \Omega_{\mathbb{G}_m} E\langle 1 \rangle \rightarrow (\Omega_{\mathbb{G}_m})^2 E\langle 2 \rangle \rightarrow \dots)$$

if each level of E is already a fibrant spectrum in Spt_{S^1} . Otherwise we can first apply the stable \mathbb{A}^1 -Nisnevich-local fibrant replacement functor $\Theta_{S^1}^{\mathbb{A}^1}$ levelwise [Hov01, Theorem 4.12]. We write $\Omega_{\mathbb{G}_m}^\infty : \text{Spt}_{\mathbb{P}^1} \rightarrow \text{Spt}_{S^1}$ for the composition of this fibrant replacement functor with $(-)_0$ from (1.1).

Preliminaries on t-structures. We briefly recall the definition of a homological t-structure and basic properties. Details can be found in [GM03].

Definition 1.7. A (homological) *t-structure* on a triangulated category \mathcal{D} is a pair of strictly (i.e., isomorphism closed) full subcategories $\mathcal{D}_{\leq 0}$ and $\mathcal{D}_{\geq 0}$ such that the following axioms hold, where for an integer n , one sets $\mathcal{D}_{\geq n} := \mathcal{D}_{\geq 0}[n]$ and $\mathcal{D}_{\leq n} := \mathcal{D}_{\leq 0}[n]$:

- For all $X \in \mathcal{D}_{\geq 0}$ and all $Y \in \mathcal{D}_{\leq -1}$ we have $\text{hom}_{\mathcal{D}}(X, Y) = 0$.
- $\mathcal{D}_{\geq 0}$ is closed under $[1]$ (i.e., $\mathcal{D}_{\geq 1} \subseteq \mathcal{D}_{\geq 0}$) and dually $\mathcal{D}_{\leq -1} \subseteq \mathcal{D}_{\leq 0}$.
- For all $Y \in \mathcal{D}$ exists a distinguished $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ with $X \in \mathcal{D}_{\geq 0}$ and $Z \in \mathcal{D}_{\leq -1}$.

Set $\mathcal{D}_{=n} := \mathcal{D}_{\geq n} \cap \mathcal{D}_{\leq n}$ and call $\mathcal{D}_{=0}$ the *heart* of the t-structure. A t-structure is called *non-degenerate* if $\cap_{n \geq 0} \mathcal{D}_{\geq n} = \{0\}$ and $\cap_{n \leq 0} \mathcal{D}_{\leq n} = \{0\}$. A t-structure is called *left-complete* if for all $X \in \mathcal{D}$ the canonical morphism

$$X \rightarrow \text{holim}_{n \rightarrow \infty} X_{\leq n}$$

is an isomorphism. Dually, a t-structure is called *right-complete* if for all $X \in \mathcal{D}$ the canonical morphism $\text{hocolim}_{n \rightarrow -\infty} X_{\geq n} \rightarrow X$ is an isomorphism.

Remark 1.8. The adjunctions

$$\tau_{\leq n} : \begin{array}{ccc} \mathcal{D}_{\geq n} & \rightleftarrows & \mathcal{D} \\ \mathcal{D} & \rightleftarrows & \mathcal{D}_{\leq n} \end{array} : \tau_{\geq n}$$

turn $\mathcal{D}_{\geq n}$ into a coreflective and $\mathcal{D}_{\leq n}$ into a reflective subcategory of \mathcal{D} . The counit of the first adjunction is denoted by $(-)_{\geq n} : \mathcal{D} \rightarrow \mathcal{D}$ and called the *skeleton*. The unit of the second adjunction is denoted by $(-)_{\leq n}$ and called the *coskeleton*. The skeleton and the coskeleton induce a distinguished triangle

$$X_{\geq n} \rightarrow X \rightarrow X_{\leq n-1} \rightarrow (X_{\geq n})[1].$$

Remark 1.9. Let \mathcal{D} be a triangulated category obtained from the homotopy category of a stable model category together with a t-structure. If the t-structure is left-complete, then $\cap_{n \geq 0} \mathcal{D}_{\geq n} = \{0\}$ which can be seen as follows: Take $X \in \cap_{n \geq 0} \mathcal{D}_{\geq n}$ and suppose $X \cong \text{holim } X_{\leq n}$. The homotopy limit of the diagram

$$\begin{array}{ccccc} & \downarrow & & \downarrow & \\ & X_{\geq n+1} & \xrightarrow{\cong} & X & \xrightarrow{\quad} & X_{\leq n} \\ & \downarrow & & \parallel & & \downarrow \\ & X_{\geq n} & \xrightarrow{\cong} & X & \xrightarrow{\quad} & X_{\leq n-1} \end{array}$$

of triangles is the triangle $\text{holim } X_{\geq n} \rightarrow X \rightarrow \text{holim } X_{\leq n}$. Since the homotopy limit of weak equivalences is a weak equivalence, one has $\text{holim } X_{\geq n} \cong X$ and hence $0 \cong \text{holim } X_{\leq n} \cong X$. In the same way, right-completeness implies $\cap_{n \leq 0} \mathcal{D}_{\leq n} = \{0\}$.

For the converse, consider [Lur16, Prop. 1.2.1.19]: Suppose that $\mathcal{D}_{\geq 0}$ is stable under countable homotopy products, then $\cap_{n \geq 0} \mathcal{D}_{\geq n} = \{0\}$ implies left-completeness. Dually, if $\mathcal{D}_{\leq 0}$ is stable under countable homotopy coproducts, the relation $\cap_{n \leq 0} \mathcal{D}_{\leq n} = \{0\}$ implies right-completeness.

Proposition 1.10 (Ayoub [Ayo07, Prop. 2.1.70]). *Let \mathcal{D} be a triangulated category with coproducts and let \mathcal{S} be a set of compact objects of \mathcal{D} . Define*

- $\mathcal{D}_{\leq -1}$ as the full subcategory of those Y of \mathcal{D} with $\text{hom}_{\mathcal{D}}(S[n], Y) = 0$ for all $n \geq 0$ and all $S \in \mathcal{S}$,
- $\mathcal{D}_{\geq 0}$ as the full subcategory of those X of \mathcal{D} with $\text{hom}_{\mathcal{D}}(X, Y) = 0$ for all $Y \in \mathcal{D}_{\leq -1}$.

The pair $\mathcal{D}_{\leq 0} = \mathcal{D}_{\leq -1}[1]$ and $\mathcal{D}_{\geq 0}$ forms a t-structure. The category $\mathcal{D}_{\geq 0}$ is the full subcategory of \mathcal{D} generated under extensions, (small) sums and cones from \mathcal{S} and in particular $\mathcal{S} \subseteq \mathcal{D}_{\geq 0}$. Moreover, the truncation functor $\tau_{\leq -1}$ is given by $\tau_{\leq -1}(X) := \text{hocolim}_{k \rightarrow \infty} \Phi^k(X)$ where $\Phi(X)$ is defined as the cone

$$\coprod_{\substack{\text{Hom}(S[n], X) \\ S \in \mathcal{S}, n \geq 0}} S[n] \rightarrow X \rightarrow \Phi(X).$$

Remark 1.11. Let \mathcal{D} be a triangulated category obtained from the homotopy category of a stable model category and let \mathcal{S} be a set of compact objects of \mathcal{D} . The t-structure obtained from the previous Proposition 1.10 satisfies the property that $\mathcal{D}_{\leq 0}$ is stable under countable homotopy coproducts. If \mathcal{D} has an underlying cofibrantly generated model category and \mathcal{S} equals (up to shifts) the set of cofibres of the generating cofibrations, then $\cap_{n \leq 0} \mathcal{D}_{\leq n} = \{0\}$ by [Hov99, Theorem 7.3.1] and \mathcal{D} is right-complete by the previous Remark 1.9. It is however usually a non-trivial issue to show left-completeness of a t-structure obtained from Proposition 1.10.

t-structures on S^1 -spectra. In this section we recall some basic properties about canonical t-structures on S^1 -spectra arising in \mathbb{A}^1 -homotopy theory.

Definition 1.12. Consider the set $\mathcal{S} := \{\Sigma_{S^1}^\infty U_+ \mid U \in \text{Sm}_S\}$. The *objectwise t-structure* (resp. *Nisnevich-local t-structure*, resp. \mathbb{A}^1 -*Nisnevich-local t-structure*) on $\mathcal{SH}_{S^1}^{ob}$ (resp. $\mathcal{SH}_{S^1}^s$, resp. $\mathcal{SH}_{S^1}^{\mathbb{A}^1}$) is obtained by applying Proposition 1.10 to the triangulated category $\mathcal{SH}_{S^1}^{ob}$ (resp. $\mathcal{SH}_{S^1}^s$, resp. $\mathcal{SH}_{S^1}^{\mathbb{A}^1}$) and to \mathcal{S} .

Remark 1.13. In the following, we take the liberty to identify homotopy categories with their equivalent subcategories of fibrant objects. This allows us for example to make sense of an intersection $\mathcal{SH}_{S^1 \leq -1}^{ob} \cap \mathcal{SH}_{S^1}^s$ even though these categories have the same objects by definition [Hov99, Def. 1.2.1].

Remark 1.14. In [Mor05] the Nisnevich-local t-structure on $\mathcal{SH}_{S^1}^s$ is called the *standard t-structure*. In [Mor04, Ch. 4.3] the \mathbb{A}^1 -Nisnevich-local t-structure on $\mathcal{SH}_{S^1}^{\mathbb{A}^1}$ is called the *homotopy t-structure* (on S^1 -spectra).

Remark 1.15. By definition we have

$$\mathcal{SH}_{S^1 \leq -1}^{ob} = \{\text{Those } Y \in \mathcal{SH}_{S^1}^{ob} \text{ with no homotopy presheaves } \pi_i Y \text{ with } i \geq 0\}.$$

Applying the classical [Mar83, Prop. 3.6] objectwise, we get

$$\mathcal{SH}_{S^1 \geq 0}^{ob} = \{\text{Those } X \in \mathcal{SH}_{S^1}^{ob} \text{ with no homotopy presheaves } \pi_i X \text{ with } i \leq -1\}.$$

The objectwise t-structure is clearly non-degenerate as there are no non-zero spectra without non-trivial homotopy presheaves. The objectwise t-structure is right-complete by Remark 1.11 and left-complete by Remark 1.9 as $\mathcal{SH}_{S^1 \geq 0}^{ob}$ is stable under countable homotopy products.

Remark 1.16. Again, by definition we have

$$\begin{aligned} \mathcal{SH}_{S^1 \leq -1}^s &= \mathcal{SH}_{S^1 \leq -1}^{ob} \cap \mathcal{SH}_{S^1}^s \\ &= \{\text{Those } Y \in \mathcal{SH}^s \text{ with no homotopy presheaves } \pi_i^s Y \text{ with } i \geq 0\} \end{aligned}$$

and using on Nisnevich-stalks the classical result [Mar83, Prop. 3.6], we get

$$\begin{aligned} \mathcal{SH}_{S^1 \geq 0}^s &= \{\text{Those } X \in \mathcal{SH}_{S^1}^s \text{ with no homotopy sheaves } \tilde{\pi}_i^s X \text{ with } i \leq -1\}, \\ \mathcal{SH}_{S^1 \leq -1}^s &= \{\text{Those } Y \in \mathcal{SH}_{S^1}^s \text{ with no homotopy sheaves } \tilde{\pi}_i^s Y \text{ with } i \geq 0\}. \end{aligned}$$

The Nisnevich-local t-structure is clearly non-degenerate as there are no non-zero spectra without non-trivial homotopy sheaves. The Nisnevich-local t-structure is right-complete by Remark 1.11 and left-complete by [Spi14, Lem. 4.4].

Remark 1.17. A Nisnevich-local fibrant replacement functor L^s respects only the truncation from above, i.e., if E is in $\mathcal{SH}_{S^1 \leq -1}^{ob}$, then the spectrum $L^s E$ is in $\mathcal{SH}_{S^1 \leq -1}^{ob} \cap \mathcal{SH}_{S^1}^s = \mathcal{SH}_{S^1 \leq -1}^s$. The analogous statement is not true for the positive part $\mathcal{SH}_{S^1 \geq 0}^{ob}$ which can be seen as follows: By Hilbert's Theorem 90, there is an isomorphism $\text{Pic}(X) \cong H_{Nis}^1(X, \mathbb{G}_m)$. The Eilenberg-MacLane spectrum $H\mathbb{G}_m$ is in the heart of the objectwise t-structure but

$$H_{Nis}^1(X, \mathbb{G}_m) = [\Sigma_{S^1}^\infty X_+, L^s H\mathbb{G}_m[1]] = [\Sigma_{S^1}^\infty X_+[-1], L^s H\mathbb{G}_m] = \pi_{-1}(L^s H\mathbb{G}_m)(X)$$

and certainly there are schemes X with non-trivial Picard group.

Remark 1.18. By definition and Remark 1.16, one has

$$\begin{aligned} \mathcal{SH}_{S^1 \leq -1}^{\mathbb{A}^1} &= \mathcal{SH}_{S^1 \leq -1}^{ob} \cap \mathcal{SH}_{S^1}^{\mathbb{A}^1} \\ &= \{\text{Those } Y \in \mathcal{SH}_{S^1}^{\mathbb{A}^1} \text{ with no homotopy presheaves } \pi_i^{\mathbb{A}^1} Y \text{ with } i \geq 0\} \\ &= \{\text{Those } Y \in \mathcal{SH}_{S^1}^{\mathbb{A}^1} \text{ with no homotopy sheaves } \tilde{\pi}_i^{\mathbb{A}^1} Y \text{ with } i \geq 0\}. \end{aligned}$$

The \mathbb{A}^1 -Nisnevich-local t-structure is right-complete by Remark 1.11 and we have $\cap_{n \leq 0} \mathcal{SH}_{S^1 \leq n}^{\mathbb{A}^1} = \{0\}$. It will be shown in Corollary 3.6 that the \mathbb{A}^1 -Nisnevich-local t-structure is left-complete and hence non-degenerate.

Definition 1.19. Following the notation of [Spi14], we define

$$\mathcal{SH}_{S^1 h \geq 0}^{\mathbb{A}^1} := \{\text{Those } X \in \mathcal{SH}_{S^1}^{\mathbb{A}^1} \text{ with no homotopy sheaves } \tilde{\pi}_i^{\mathbb{A}^1} X \text{ with } i \leq -1\}.$$

Remark 1.20. The full subcategory $\mathcal{SH}_{S^1 h \geq 0}^{\mathbb{A}^1} \subseteq \mathcal{SH}_{S^1}^{\mathbb{A}^1}$ is closed under homotopy colimits and extensions. There is an inclusion $\mathcal{SH}_{S^1 h \geq 0}^{\mathbb{A}^1} \subseteq \mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1}$ [Spi14, Lem. 4.1, 4.3]. The other implication $\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1} \subseteq \mathcal{SH}_{S^1 h \geq 0}^{\mathbb{A}^1}$ holds if and only if $L^{\mathbb{A}^1} \Sigma_{S^1}^{\infty} U_+ \in \mathcal{SH}_{S^1 h \geq 0}^{\mathbb{A}^1}$ for all $U \in \text{Sm}_S$. Unfortunately, there are schemes S , such that these two equivalent conditions do not hold (see Remark 4.7). However, they hold true over the spectrum of a field S [Mor05, Theorem 6.1.8].

t-structures on \mathbb{P}^1 -spectra. In this section we recall the homotopy t-structure on the motivic homotopy category \mathcal{SH} . We remind the reader that $\langle q \rangle$ denotes the \mathbb{G}_m -shift operation.

Definition 1.21. The *homotopy t-structure* on \mathcal{SH} is obtained by applying Proposition 1.10 to the triangulated category \mathcal{SH} and the set

$$\mathcal{S} = \{\Sigma_{\mathbb{P}^1}^{\infty}(U_+) \langle q \rangle \mid U \in \text{Sm}_S \text{ and } q \in \mathbb{Z}\}.$$

Remark 1.22. We use the name “homotopy t-structure” in order to agree with the terminology of [Mor04] and [Mor05].

Remark 1.23. Unravelling the definitions, one identifies

$$\begin{aligned} \mathcal{SH}_{\leq -1} &= \{\text{Those } Y \in \mathcal{SH} \text{ with } \Omega_{\mathbb{G}_m}^{\infty}(Y \langle q \rangle) \in \mathcal{SH}_{S^1 \leq -1}^{\mathbb{A}^1} \text{ for all } q \in \mathbb{Z}\} \\ &= \{\text{Those } Y \in \mathcal{SH} \text{ with } (\text{colim}_k \Omega_{\mathbb{G}_m}^k Y_{k+q}) \in \mathcal{SH}_{S^1 \leq -1}^{\mathbb{A}^1} \text{ for all } q \in \mathbb{Z}\}. \end{aligned}$$

In particular

$$\begin{aligned} \Omega_{\mathbb{G}_m}^{\infty}(\mathcal{SH}_{\leq -1}) &\subseteq \mathcal{SH}_{S^1 \leq -1}^{\mathbb{A}^1} \text{ and} \\ \Sigma_{\mathbb{G}_m}^{\infty}(\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1}) &\subseteq \mathcal{SH}_{\geq 0}, \end{aligned}$$

using [Ayo06, Lem. 2.1.16] for the latter.

Remark 1.24. Over a field, using [Mor04, Lem. 4.3.11] and the equality $\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1} = \mathcal{SH}_{S^1 h \geq 0}^{\mathbb{A}^1}$ from Remark 1.20, we can also identify

$$\mathcal{SH}_{\geq 0} = \{\text{Those } X \in \mathcal{SH} \text{ with } \Omega_{\mathbb{G}_m}^{\infty}(X \langle q \rangle) \in \mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1} \text{ for all } q \in \mathbb{Z}\}$$

(cf. [Mor04, Ch. 5.2]). In particular, we have $\Omega_{\mathbb{G}_m}^{\infty}(\mathcal{SH}_{\geq 0}) \subseteq \mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1}$ in this case.

Remark 1.25. The homotopy t-structure on the motivic homotopy category is right-complete by Remark 1.11 and we have $\bigcap_{n \leq 0} \mathcal{SH}_{\leq n} = \{0\}$. It will be shown in Corollary 3.8 that the homotopy t-structure on the motivic homotopy category is also left-complete and hence non-degenerate.

2. GABBER-PRESENTATIONS OVER HENSELIAN DISCRETE VALUATION RINGS

Throughout the whole section, fix a henselian discrete valuation ring \mathfrak{o} with maximal ideal $\mathfrak{m} \trianglelefteq \mathfrak{o}$, local uniformizer $\pi \in \mathfrak{m}$, residue field $\mathbb{F} = \mathfrak{o}/\mathfrak{m}$ and field of fractions k . Assume that \mathbb{F} is an infinite field. Let S be the spectrum of \mathfrak{o} and denote by σ resp. η the closed resp. generic point of S . We want to prove the following version of Gabber’s geometric presentation lemma over \mathfrak{o} :

Theorem 2.1. *Let \mathfrak{o} be a henselian discrete valuation ring with infinite residue field. Let X/\mathfrak{o} be a smooth \mathfrak{o} -scheme of finite type and let $Z \hookrightarrow X$ be a proper closed subscheme. Let z be a point in Z . If z lies in the special fibre, suppose that $Z_{\sigma} \neq X_{\sigma}$. Then Nisnevich-locally around z there is an étale map $p: X \rightarrow \mathbb{A}_V^1$ with $V \in \text{Sm}_S$ s.t. Z is finite over V and*

$$\begin{array}{ccc} X \setminus Z & \xrightarrow{\quad} & X \\ \downarrow & & \downarrow p \\ \mathbb{A}_V^1 \setminus p(Z) & \xrightarrow{\quad} & \mathbb{A}_V^1 \end{array}$$

is a distinguished square in the smooth Nisnevich site over \mathfrak{o} . In particular, the canonical morphism $X/(X \setminus Z) \rightarrow \mathbb{A}_V^1/(\mathbb{A}_V^1 \setminus p(Z))$ is an isomorphism of Nisnevich sheaves.

Linear projections. Denote by $\mathbb{A}_{x_1/x_0, \dots, x_N/x_0, S} = \mathbb{A}_{\underline{x}/x_0, S}$ resp. $\mathbb{P}_{x_0: \dots: x_N, S} = \mathbb{P}_{\underline{x}, S}$ the affine resp. projective N -space \mathbb{A}_S^N resp. \mathbb{P}_S^N with coordinates $\frac{x_1}{x_0}, \dots, \frac{x_N}{x_0}$ resp. homogeneous coordinates $x_0 : \dots : x_N$. We get the standard open embedding $\mathbb{A}_{\underline{x}/x_0, S} \hookrightarrow \mathbb{P}_{\underline{x}, S}$. By abuse of notation, we identify $\frac{x_1}{x_0}, \dots, \frac{x_N}{x_0}$ with x_1, \dots, x_N and write just $\mathbb{A}_{\underline{x}, S}$ for $\mathbb{A}_{\underline{x}/x_0, S}$ (and similarly for other coordinates).

Let \mathfrak{E} be the dual affine space $\mathbb{A}_{x_1^*, \dots, x_N^*, S}^\vee = \mathbb{A}_{x_1^*, \dots, x_N^*, S}$, i.e., for any \mathfrak{o} -Algebra A , $\mathfrak{E}(A)$ is the free A -module generated by the coordinate functions x_1, \dots, x_N of $\mathbb{A}_{\underline{x}, S}$. To be more precise, take r copies of \mathfrak{E} and denote the coordinates of the j -th-copy $\mathfrak{E}^{(j)}$ by $x_{1,j}^*, \dots, x_{N,j}^*$. Mapping $t_j \mapsto \sum_i x_i \otimes x_{i,j}^*$ defines the dual pairing $\mathbb{A}_{\underline{x}, S} \times_S \mathfrak{E}^{(j)} \rightarrow \mathbb{A}_{t_j, S}$. Let A be an \mathfrak{o} -algebra and \underline{u} an A -point of $\mathfrak{E}^{\times s^r} = \prod_{1 \leq j \leq r, S} \mathfrak{E}^{(j)}$. Using the above pairings, \underline{u} induces a linear A -morphism

$$\underline{u}: \mathbb{A}_{\underline{x}, A} \longrightarrow \mathbb{A}_{t_1, \dots, t_r, A} \quad \text{via } t_j \longmapsto \sum_i u_j(x_{i,j}^*)x_i.$$

Mapping $t_0 \mapsto x_0$, this extends to a rational map

$$\underline{u}: \mathbb{P}_{\underline{x}, A} \dashrightarrow \mathbb{P}_{t_0: \dots: t_r, S}$$

with center $L_{\underline{u}} := V_+(x_0, u_1, \dots, u_r) \subseteq H_\infty$, where $H_\infty \subset \mathbb{P}_{\underline{x}, A}$ is the hyperplane at infinity $V_+(x_0)$.

Say $Y \hookrightarrow \mathbb{A}_{\underline{x}, A}$ is a (reduced) closed subscheme with (reduced) projective closure $\bar{Y} \hookrightarrow \mathbb{P}_{\underline{x}, A}$ and assume $\bar{Y} \cap L_{\underline{u}} = \emptyset$. Then \underline{u} induces regular maps

$$p_{\underline{u}}: Y \longrightarrow \mathbb{A}_{t_1, \dots, t_r, A} \quad \text{and} \quad \bar{p}_{\underline{u}}: \bar{Y} \longrightarrow \mathbb{P}_{t_0: \dots: t_r, A}$$

satisfying $p_{\underline{u}} = \bar{p}_{\underline{u}} \times_{\mathbb{P}_{\underline{x}, A}} \mathbb{A}_{\underline{x}, A}$. Observe that [Sha94, Theorem. I.5.3.7] remains true in our setting:

Lemma 2.2. *For any $\underline{u} \in \mathfrak{E}^{\times s^r}(A)$ and any closed $Y \hookrightarrow \mathbb{A}_{\underline{x}, S}$ with $\bar{Y} \cap L_{\underline{u}} = \emptyset$, the linear projections $p_{\underline{u}}$ and $\bar{p}_{\underline{u}}$ are finite maps.*

Proof. It suffices to show that $\bar{p}_{\underline{u}}$ is finite. As a map between projective schemes over S , $\bar{p}_{\underline{u}}$ itself is projective. It remains to show that $\bar{p}_{\underline{u}}$ is quasi-finite: Let $\bar{\sigma}$ resp. $\bar{\eta}$ be a geometric point of S over σ resp. η . By [Sha94, Theorem. I.5.3.7], $\bar{\sigma}^* \bar{p}_{\underline{u}}$ resp. $\bar{\eta}^* \bar{p}_{\underline{u}}$ is finite. It follows that $\bar{p}_{\underline{u}}$ is finite on the special resp. geometric fibre $\sigma^* \bar{p}_{\underline{u}}$ resp. $\eta^* \bar{p}_{\underline{u}}$, hence quasi-finite. \square

Gabber's geometric presentation lemma. Theorem 2.1 is a consequence of the following version of [CTHK97, Theorem. 3.2.2] over \mathfrak{o} :

Theorem 2.3. *Let $X = \text{Spec}(A)/S$ be a smooth affine S -scheme of finite type, fibrewise of pure dimension n and let $Z = \text{Spec}(B) \hookrightarrow X$ be a proper closed subscheme. Let z be a point in Z . If z lies in the special fibre, suppose that $Z_\sigma \neq X_\sigma$. Then, Nisnevich-locally around z , there exists a closed embedding $X \hookrightarrow \mathbb{A}_S^N$ and a Zariski-open subset $W \subseteq \mathfrak{E}^{\times s^n}$ with $W(\mathfrak{o}) \neq \emptyset$, s.t. the following holds:*

For all $\underline{u} \in W(\mathfrak{o})$ with linear projections $p_{\underline{u}} = p_{(u_1, \dots, u_{n-1})} \times_S p_{u_n}: X \rightarrow \mathbb{A}_S^n = \mathbb{A}_S^{n-1} \times_S \mathbb{A}_S^1$ there are Zariski-open neighbourhoods $p_{(u_1, \dots, u_{n-1})}(z) \in V \subseteq \mathbb{A}_S^{n-1}$ and $z \in U \subseteq p_{(u_1, \dots, u_{n-1})}^{-1}(V)$ satisfying:

- (1) $p_{(u_1, \dots, u_{n-1})}|_Z: Z \rightarrow \mathbb{A}_S^{n-1}$ is finite,
- (2) $Z \cap U = Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V)$,
- (3) $p_{\underline{u}}|_U: U \rightarrow \mathbb{A}_S^n$ is étale and restricts to a closed embedding $Z \cap U \rightarrow \mathbb{A}_V^1$ and
- (4) $p_{\underline{u}}^{-1}(p_{\underline{u}}(Z \cap U)) \cap U = Z \cap U$.

The proof of Theorem 2.3 will follow the proof in [CTHK97]. Let us first make a few easy reductions:

Reduction 2.4. Clearly, we may assume that both X and Z are connected. Next, observe that the case of z lying in the generic fibre X_η of X/S is already covered by [CTHK97, Theorem 3.2.2]. Thus, we may assume that z lies in the special fibre X_σ of X/S . Finally, observe that we may enlarge Z . In particular, picking any element f in the kernel of $A \twoheadrightarrow B$ with $f \neq 0$ in the special fibre $A \otimes_{\mathfrak{o}} \mathbb{F}$, we may assume $B = A/f$, i.e., $Z = V(f)$.

Towards the finiteness part. The key part in the proof of Theorem 2.3 is the finiteness assertion (1). By Lemma 2.2, we need to find a closed embedding $i_0: X \hookrightarrow \mathbb{A}_{\underline{x}, S}$ and an \mathfrak{o} -point $\underline{u} \in \mathfrak{E}^{\times s^n}(\mathfrak{o})$ s.t. the closure \bar{Z} of Z in $\mathbb{P}_{\underline{x}, S}$ intersects $L_{(u_1, \dots, u_{n-1})}$ trivially. Unfortunately, it is not enough to require that the fibrewise closure of Z in $\mathbb{P}_{\underline{x}, S}$ misses $L_{(u_1, \dots, u_{n-1})}$. Indeed, \bar{Z} might not be the fibrewise projective closure of Z over S :

Example 2.5. Let $A = \mathfrak{o}[x_1]$ and $B = \mathfrak{o}[x_1]/(\pi x_1^2 + x_1 + 1)$. Then $\text{Spec}(B) \subset \mathbb{P}_{x_0: x_1, S}$ is fibrewise closed but at least one solution of $\pi x_1^2 + x_1 + 1$ in k^{alg} specializes to ∞ in $\mathbb{P}_{x_0: x_1, \mathbb{F}}$, i.e., $\text{Spec}(B) \subset \mathbb{P}_{x_0: x_1, S}$ is not closed.

To avoid these difficulties, we need to make a careful choice for the embedding $i_0: X \hookrightarrow \mathbb{A}_{\underline{x}, S}$:

Proposition 2.6. *Nisnevich-locally around z , there exists a closed embedding $i_0: X \hookrightarrow \mathbb{A}_{\underline{x}, S}$ with Z fibrewise dense over S inside its closure \bar{Z} in $\mathbb{P}_{\underline{x}, S}$.*

Proof. We need to adapt [Kai15, Theorem 4.6] to our situation. Since z lies in the special fibre and \mathfrak{o} is henselian, loc.cit. gives us an affine Nisnevich-neighbourhood $(Z', z') \rightarrow (Z, z)$ and a closed embedding $Z' \hookrightarrow \mathbb{A}_{x_1, \dots, x_m, S}$, s.t. Z' is fibrewise dense over S inside its closure \bar{Z}' in $\mathbb{P}_{x_0: \dots, x_m, S}$. Unfortunately, refining (Z', z') by an arbitrary Zariski-neighbourhood may destroy the fibrewise density. But looking at the proof of loc.cit., we can get a slightly stronger statement:

Using a suitable linear projection, Kai expresses Z (X in loc.cit.) as a quasi-projective relative curve $Z \rightarrow T$ for T a suitable smooth S -scheme. Say z maps to $t \in T$. Then all Nisnevich-localizations of (Z, z) occurring in the proof of loc.cit. are either Zariski-localizations or are pullbacks from Nisnevich-localizations of (T, t) . The latter Nisnevich-localizations are in turn either Zariski-localizations, may be taken as standard étale maps or are given by the induction hypothesis applied to (T, t) . The induction basis (Z has fibres of dimension 0) can be handled using Zariski-localizations only. Summing up, we may even assume that Z' is an affine Zariski-neighbourhood of z' in $Z'' = \text{Spec}(B'')$ with $B'' = B[t_1, \dots, t_s]/(\bar{g}_1, \dots, \bar{g}_s)$ and invertible Jacobi-determinant $\det(\{\frac{d\bar{g}_i}{dt_j}\}_{i,j}) \in B''^\times$.

We want to extend (Z', z') to a Nisnevich-neighbourhood of z in X' : Since the Jacobi-determinant $\det(\{\frac{d\bar{g}_i}{dt_j}\}_{i,j})$ is invertible in B'' , it is non-trivial in $B'' \otimes k(z')$. Choose a lift $g_i \in A[\underline{t}]$ for each \bar{g}_i and set $A'' := A[\underline{t}]/(g_1, \dots, g_s)$ and $X'' = \text{Spec}(A'')$. By construction, $B'' = A'' \otimes_A B$, so z' induces a point (also denoted by) z'' in X'' . Since $\det(\{\frac{dg_i}{dt_j}\}_{i,j}) \equiv \det(\{\frac{d\bar{g}_i}{dt_j}\}_{i,j}) \neq 0$ in $A'' \otimes k(z')$, the Jacobi-determinant $\det(\{\frac{dg_i}{dt_j}\}_{i,j})$ is invertible around z'' in X'' . By shrinking X'' (without changing Z'' , since the Jacobi-determinant is invertible on the latter), we may assume that $(Z'', z'') \rightarrow (Z, z)$ extends to an affine Nisnevich-neighbourhood $(X'', z'') \rightarrow (X, z)$. Since Z' is an open affine neighbourhood of z in Z'' , we may further shrink X'' to get the desired affine Nisnevich-neighbourhood $(X', z') \rightarrow (X, z)$ inducing Z' over Z . Further, lifting the images of x_i in $B' = \mathcal{O}(Z')$ to $A' = \mathcal{O}(X')$, i extends to a map $i': X' \rightarrow \mathbb{A}_{x_1, \dots, x_m, S}$.

Unfortunately, there is no reason for i' to be a closed embedding. To repair this, choose a closed embedding $X' \hookrightarrow \mathbb{A}_{y_1, \dots, y_r, S}$ over S , i.e., generators \bar{y}_i of A' as an \mathfrak{o} -algebra. Writing $\bar{y}^{\underline{i}} = \bar{y}_1^{i_1} \cdots \bar{y}_r^{i_r}$, any element of the ideal $f \cdot A'$ is of the form $\sum_{\underline{i}} a_{\underline{i}} \cdot f \bar{y}^{\underline{i}}$ where $a_{\underline{i}} \in \mathfrak{o}$ and \underline{i} runs through a finite subset of \mathbb{N}^r . Mapping $y^{\underline{i}} \mapsto f \bar{y}^{\underline{i}}$, we get a map $X' \rightarrow \mathbb{A}_{\{y^{\underline{i}} | \underline{i} \in \mathbb{N}^r\}, S}$ into a copy of the infinite affine space over S . Combining this map with i' , we get a closed embedding $i_\infty: X' \hookrightarrow \mathbb{A}_{x_1, \dots, x_m, S} \times_S \mathbb{A}_{\{y^{\underline{i}} | \underline{i} \in \mathbb{N}^r\}, S} \cong \mathbb{A}_S^\infty$: Indeed, $i': \mathfrak{o}[x_1, \dots, x_m] \rightarrow A'$ is surjective modulo f and $\mathfrak{o}[y^{\underline{i}} | \underline{i} \in \mathbb{N}^r] \rightarrow A'$ has image $\mathfrak{o}[f \cdot A']$ by construction. By Lemma 2.7, below, i_∞ induces $i_0: X' \hookrightarrow \mathbb{A}_{x_1, \dots, x_m, S} \times_S \mathbb{A}_{y^{\underline{i}_1}, \dots, y^{\underline{i}_l}, S}$ still a closed embedding for $\underline{i}_1, \dots, \underline{i}_l \in \mathbb{N}^r$ suitable. Setting $x_{m+j} := y^{\underline{i}_j}$ and $N = m + l$, we have constructed a closed embedding $i_0: X' \hookrightarrow \mathbb{A}_{x_1, \dots, x_N, S}$ s.t. $i_0|_{Z'}$ factors over $i: Z' \hookrightarrow \mathbb{A}_{x_1, \dots, x_m, S} = V(x_{m+1}, \dots, x_N)$. In particular, the closure of Z' in $\mathbb{P}_{x_0: \dots, x_N, S}$ is just \bar{Z}' inside the linear subspace $V_+(x_{m+1}, \dots, x_N)$, so Z' is fibrewise dense over S inside this closure. \square

Lemma 2.7. *Let C be an \mathfrak{o} -algebra of finite type. Let $\iota: \text{Spec}(C) \hookrightarrow \mathbb{A}_{t_1, t_2, \dots, S} = \mathbb{A}_S^\infty$ be a closed embedding and let $\text{pr}_{\leq N}: \mathbb{A}_{t_1, t_2, \dots, S} \rightarrow \mathbb{A}_{t_1, \dots, t_N, S} = \mathbb{A}_S^N$ be the canonical projection. Then $\text{pr}_{\leq N} \circ \iota$ is a closed embedding for $N \gg 0$.*

Proof. Say, as an \mathfrak{o} -algebra, C is generated by $c_1, \dots, c_r \in C$. Since the corresponding map on algebras $\mathfrak{o}[t_1, t_2, \dots] \rightarrow C$ is surjective, we can find polynomials $f_i \in \mathfrak{o}[t_1, t_2, \dots]$ mapping to c_i . Pick $N \gg 0$ s.t. all the f_i lie inside $\mathfrak{o}[t_1, \dots, t_N]$. Then ι restricted to $\mathfrak{o}[t_1, \dots, t_N]$ is still surjective, hence the claim. \square

Choosing linear projections. In the next step of the proof of Theorem 2.3, we want to find the Zariski-open subset $W \subseteq \mathfrak{E}^{\times s^n}$ parametrizing the linear projections $p_{\underline{u}}$. To do so, let us first make one further reduction:

Reduction 2.8. By Proposition 2.6, we may assume that X admits a closed embedding $i_0: X \hookrightarrow \mathbb{A}_{\underline{x}, S}$ s.t. Z is fibrewise dense over S inside its closure \bar{Z} in $\mathbb{P}_{\underline{x}, S}$. Having made this reduction, we may replace z by any closed point z' inside the closure of $\{z\}$, i.e., we may assume that z is a closed point in the following.

The Zariski-open subset $W \subseteq \mathfrak{E}^{\times s^n}$ in Theorem 2.3 will be provided in the following proposition:

Proposition 2.9. *Let $X = \text{Spec}(A)/S$ be a connected smooth affine S -scheme of finite type, fibrewise of pure dimension n , f an element in A which is non-zero in $A \otimes_{\mathfrak{o}} \mathbb{F}$ and $Z = \text{Spec}(B = A/f) \hookrightarrow X$ the closed embedding. Let z be a closed point in the special fibre of Z . Suppose there is a closed embedding $i_0: X \hookrightarrow \mathbb{A}_{\underline{x}, S}$ s.t. Z is fibrewise dense over S inside its closure \bar{Z} in $\mathbb{P}_{\underline{x}, S}$.*

Then there is a Zariski-open subset $W \subseteq \mathfrak{E}^{\times sn}$ with $W(\mathfrak{o}) \neq \emptyset$, s.t. for all $\underline{u} \in W(\mathfrak{o})$ the following holds:

- (1) $p_{(u_1, \dots, u_{n-1})}|_Z: Z \rightarrow \mathbb{A}_{t_1, \dots, t_{n-1}, S}$ is finite,
- (2) $p_{\underline{u}}$ is étale at all points of $F := p_{(u_1, \dots, u_{n-1})}^{-1}(p_{(u_1, \dots, u_{n-1})}(z)) \cap Z$ and
- (3) $p_{\underline{u}}|_F: F \rightarrow p_{\underline{u}}(F)$ is radicial.

Before giving the proof of Proposition 2.9, let us first fix the following notation:

Remark 2.10. For Y/S a smooth scheme, denote by $\text{red}: Y(\mathfrak{o}) \rightarrow Y(\mathbb{F}) = Y_{\sigma}(\mathbb{F})$ the reduction map we get by pre-composing with the closed point σ . Because \mathfrak{o} is henselian and Y/S smooth, this reduction map is always surjective.

Proposition 2.9 follows by choosing W as the intersection of the subsets W_1 and W_2 of $\mathfrak{E}^{\times sn}$ provided by the following two lemmas. Note, that $W(\mathfrak{o})$ is non-empty: Indeed, the reduction map $W(\mathfrak{o}) \rightarrow W(\mathbb{F})$ is surjective and $W_1(\mathbb{F}) \cap W_2(\mathbb{F})$ is non-empty as the special fibre of $W_1 \cap W_2$ is a non empty open subscheme of an affine space over the infinite field \mathbb{F} .

Lemma 2.11. *Under the assumptions of Proposition 2.9, there is a Zariski-open subset $W_1 \subseteq \mathfrak{E}^{\times sn}$ with $W_1(\mathfrak{o}) \neq \emptyset$, s.t. for all $\underline{u} \in W_1(\mathfrak{o})$ the restriction $p_{(u_1, \dots, u_{n-1})}|_Z: Z \rightarrow \mathbb{A}_{t_1, \dots, t_{n-1}, S}$ is finite.*

Lemma 2.12. *Under the assumptions of Proposition 2.9, there is a Zariski-open subset $W_2 \subseteq \mathfrak{E}^{\times sn}$ with $W_2(\mathfrak{o}) \neq \emptyset$, s.t. $p_{\underline{u}}$ is étale at all points of F and $p_{\underline{u}}|_F: F \rightarrow p_{\underline{u}}(F)$ is radicial for all $\underline{u} \in W_2(\mathfrak{o})$.*

Proof of Lemma 2.11. This is just a version of the arguments leading to [Gra78, Prop. 1.1]: Recall that the j^{th} -factor $\mathfrak{E}^{(j)}$ of $\mathfrak{E}^{\times sn}$ is $\mathbb{A}_{x_{1,j}^*, \dots, x_{n,j}^*, S}$, i.e., $\mathfrak{E}^{\times sn} = \mathbb{A}_{\{x_{i,j}^* | 1 \leq i \leq n, 1 \leq j \leq n\}, S}$. Define $\mathbb{L} := V_+(x_0, \sum_j x_{i,j}^* \otimes x_j | 1 \leq i < n) \subseteq \mathfrak{E}^{\times sn} \times_S H_{\infty}$ and $\bar{Z}_{\infty} := \bar{Z} \cap H_{\infty}$ (here, $H_{\infty} = V_+(x_0) \subset \mathbb{P}_{\underline{x}, S}$ is the hyperplane at infinity). By construction, $\mathbb{L} \rightarrow \mathfrak{E}^{\times sn}$ has fibre $\mathbb{L}_{\underline{u}} = L_{(u_1, \dots, u_{n-1})}$ over $\underline{u} \in \mathfrak{E}^{\times sn}(\mathfrak{o})$. Since the projection $\text{pr}: \mathfrak{E}^{\times sn} \times_S H_{\infty} \rightarrow \mathfrak{E}^{\times sn}$ is projective, hence closed,

$$W_1 := \mathfrak{E}^{\times sn} \setminus \text{pr}(\mathbb{L} \cap (\mathfrak{E}^{\times sn} \times_S \bar{Z}_{\infty}))$$

is open. Again by construction, for any $\underline{u} \in W_1(\mathfrak{o})$, $L_{(u_1, \dots, u_{n-1})} \cap \bar{Z} = \emptyset$, so $p_{(u_1, \dots, u_{n-1})}|_Z: Z \rightarrow \mathbb{A}_{t_1, \dots, t_{n-1}, S}$ is finite by Lemma 2.2.

It remains to show that $W_1(\mathfrak{o}) \neq \emptyset$: The reduction map $\text{red}: W_1(\mathfrak{o}) \rightarrow W_1(\mathbb{F})$ is surjective, so we have to show $W_1(\mathbb{F}) = W_{1,\sigma}(\mathbb{F}) \neq \emptyset$. The special fibre $W_{1,\sigma}$ equals $\mathfrak{E}_{\sigma}^{\times sn} \setminus \text{pr}(\mathbb{L}_{\sigma} \cap (\mathfrak{E}_{\sigma}^{\times sn} \times_{\mathbb{F}} \bar{Z}_{\infty, \sigma}))$. Further, $Z_{\sigma} \subset \bar{Z}_{\sigma}$ is dense by assumption so \bar{Z}_{σ} is the closure $\overline{Z_{\sigma}}$ of Z_{σ} inside $\mathbb{P}_{\underline{x}, \mathbb{F}}$. It follows that $\bar{Z}_{\infty, \sigma} = \overline{Z_{\sigma}} \cap H_{\infty, \sigma}$, i.e., we are in the situation of [Gra78, Prop. 1.1] and $W_{1,\sigma}(\mathbb{F}) \neq \emptyset$. \square

Lemma 2.12 can easily be derived from [CTHK97, Lem. 3.4.1, Lem. 3.4.2] applied over the special fibre:

Proof of Lemma 2.12. As a closed embedding of smooth S -schemes, $i_0: X \hookrightarrow \mathbb{A}_{\underline{x}, S}$ is regular. Thus, $p_{\underline{u}}$ is flat as a complete intersection and $p_{\underline{u}}$ is étale at a point $x \in F$ if and only if $\Omega_{X/\mathbb{A}_{\underline{x}, S}}^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{X, x}$ is trivial. Now

$$\Omega_{\mathbb{A}_{\underline{x}, S}/S}^1 \otimes_{\mathfrak{o}[\underline{x}]} \mathcal{O}_{X, x} \rightarrow \Omega_{X/S}^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{X, x} \rightarrow \Omega_{X/\mathbb{A}_{\underline{x}, S}}^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{X, x} \rightarrow 0$$

is an exact sequence and the left-hand and middle terms are free $\mathcal{O}_{X, x}$ -modules of rank n . Pick the standard basis $\{dt_i \otimes 1\}_i$ of $\Omega_{\mathbb{A}_{\underline{x}, S}/S}^1 \otimes_{\mathfrak{o}[\underline{x}]} \mathcal{O}_{X, x}$. Unravelling the definitions, an element $dt_i \otimes 1$ maps to $d\bar{u}_i \otimes 1$, where \bar{u}_i is the image of the linear polynomial $u_i \in \mathfrak{o}[\underline{x}]$ in A . By Nakayama, the right-hand term $\Omega_{X/\mathbb{A}_{\underline{x}, S}}^1 \otimes_{\mathcal{O}_X} \mathcal{O}_{X, x}$ is trivial if and only if $\{d\bar{u}_i\}_i$ forms a $k(x)$ -basis of $\Omega_{X/S}^1 \otimes_{\mathcal{O}_X} k(x)$. Since $\Omega_{X/S}^1$ is free of rank n , this is equivalent to $d\bar{u}_1 \wedge \dots \wedge d\bar{u}_n \neq 0$ in $\Omega_{X/S}^n \otimes_{\mathcal{O}_X} k(x)$. But x lies in the special fibre (since z does), so $\Omega_{X/S}^n \otimes_{\mathcal{O}_X} k(x) = \Omega_{X_{\sigma}/\mathbb{F}}^n \otimes_{\mathcal{O}_{X_{\sigma}}} k(x)$. Repeating everything over the special fibre we get that $p_{\underline{u}}$ is étale at $x \in F$ if and only if $p_{\underline{u}, \sigma} = p_{\text{red}(\underline{u})}$ is étale at x .

Further, $p_{\underline{u}}|_F = p_{\text{red}(\underline{u})}|_F$, so we are in fact in the situation of [CTHK97]. Thus, by loc.cit. Lemma 3.4.1 and 3.4.2 combined with the arguments leading to Lemma 3.4.3 of, there is an open subset $\bar{W}_2 \subseteq \mathfrak{E}_{\sigma}^{\times sn}$ with $\bar{W}_2(\mathbb{F}) \neq \emptyset$, s.t. $p_{\underline{u}}$ is étale at all points of F and $p_{\underline{u}}|_F$ is radicial for all $\underline{u} \in \bar{W}_2(\mathbb{F})$. Let $W_2 \subseteq \mathfrak{E}^{\times sn}$ be any open subset with special fibre $W_{2,\sigma} = \bar{W}_2$. Then $\underline{u} \in W_2(\mathfrak{o})$ if and only if $\text{red}(\underline{u}) \in \bar{W}_2(\mathbb{F})$ and $W_2(\mathfrak{o}) \neq \emptyset$, since the reduction map is surjective. \square

Choosing neighbourhoods. Fix a linear projection $p_{\underline{u}}$ for an \mathfrak{o} -point \underline{u} in the open subset $W \subseteq \mathfrak{E}^{\times sn}$ provided by Proposition 2.9. In the following, we will construct the open neighbourhoods V and U in Theorem 2.3.

As in [CTHK97], we will first secure $V \subseteq \mathbb{A}_{t_1, \dots, t_{n-1}, S}$ in Lemma 2.13 and an open neighbourhood $z \in U_1 \subseteq p_{(u_1, \dots, u_{n-1})}^{-1}(V)$ covering Theorem 2.3 parts (2) and (4) in Lemma 2.14. The proofs can almost literally be transferred from [CTHK97].

Lemma 2.13. (cf. [CTHK97, Lem. 3.5.1]) *Under the assumptions of Proposition 2.9, there is an open neighbourhood $V \subseteq \mathbb{A}_{t_1, \dots, t_{n-1}, S}$ of $p_{(u_1, \dots, u_{n-1})}(z)$ such that the linear projection $p_{\underline{u}}$ is étale at all points of $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V)$ and the restriction $p_{\underline{u}}|_{Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V)}: Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V) \rightarrow \mathbb{A}_V^1$ is a closed embedding.*

Proof. We will get V as $V_1 \cap V_2$, where $V_1 \subseteq \mathbb{A}_{t_1, \dots, t_{n-1}, S}$ is an open neighbourhood of $p_{(u_1, \dots, u_{n-1})}(z)$ s.t. $p_{\underline{u}}$ is étale at all points of $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V_1)$ and $V_2 \subseteq \mathbb{A}_{t_1, \dots, t_{n-1}, S}$ is an open neighbourhood such that $p_{\underline{u}}$ restricts to a closed embedding $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V_2) \rightarrow \mathbb{A}_{V_2}^1$. Let $U' \subseteq X$ be the étale locus of $p_{\underline{u}}$. Since $\underline{u} \in W(\mathfrak{o})$, U' is an open neighbourhood of F in X . As $p_{(u_1, \dots, u_{n-1})}|_Z$ is finite, $p_{(u_1, \dots, u_{n-1})}(Z \setminus U')$ is closed and we set

$$V_1 := \mathbb{A}_{t_1, \dots, t_{n-1}, S} \setminus p_{(u_1, \dots, u_{n-1})}(Z \setminus U').$$

By construction, V_1 is an open neighbourhood of the image $p_{(u_1, \dots, u_{n-1})}(z)$ of z and $Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V_1) \subseteq Z \cap U'$ is contained in the étale locus U' of $p_{\underline{u}}$.

To get the neighbourhood V_2 , consider $p_{\underline{u}}|_Z: Z \rightarrow \mathbb{A}_{\underline{t}, S} = \mathbb{A}_{t_n, \mathbb{A}_{t_1, \dots, t_{n-1}, S}}$ as a family of maps over $\mathbb{A}_{t_1, \dots, t_{n-1}, S}$. Since $Z/\mathbb{A}_{t_1, \dots, t_{n-1}, S}$ is finite, the property “ $p_{\underline{u}}|_Z$ is a closed embedding” is Zariski-open in the base $\mathbb{A}_{t_1, \dots, t_{n-1}, S}$ by Nakayama. Thus we have to show that the fibre

$$p_{\underline{u}}|_F: Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(p_{(u_1, \dots, u_{n-1})}(z)) = F \rightarrow \mathbb{A}_{t_n, p_{(u_1, \dots, u_{n-1})}(z)}$$

over $p_{(u_1, \dots, u_{n-1})}(z)$ is a closed embedding. But $p_{\underline{u}}(F) \subset \mathbb{A}_{t_n, p_{(u_1, \dots, u_{n-1})}(z)}$ is closed as a finite set of closed points and $p_{\underline{u}}|_F: F \rightarrow p_{\underline{u}}(F)$ is a closed embedding as it is radicial and $p_{\underline{u}}$ is étale hence unramified at each point of F . \square

Lemma 2.14. (cf. [CTHK97, Lem. 3.6.1]) *Under the assumptions of Proposition 2.9, let $Z' := Z \cap p_{(u_1, \dots, u_{n-1})}^{-1}(V)$ and $U_1 := p_{(u_1, \dots, u_{n-1})}^{-1}(V) \setminus (p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \setminus Z')$. Then $U_1 \subseteq p_{(u_1, \dots, u_{n-1})}^{-1}(V)$ is an open neighbourhood of the point z , $Z \cap U_1 = Z'$ and $p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \cap U_1 = Z'$.*

Proof. By definition of U_1 , z lies inside U_1 , $Z \cap U_1 = Z'$ and $p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \cap U_1 = Z'$. It remains to show that $U_1 \subseteq p_{(u_1, \dots, u_{n-1})}^{-1}(V)$ is open. By Lemma 2.13, $p_{\underline{u}}$ restricts to a closed embedding $Z' \rightarrow \mathbb{A}_{t_n, V}$, so $p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \subset p_{(u_1, \dots, u_{n-1})}^{-1}(V)$ is closed. We need to show that $Z' \subseteq p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))$ is open.

To show this, consider the étale locus U'' of $p_{\underline{u}}|_{p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))}: p_{\underline{u}}^{-1}(p_{\underline{u}}(Z')) \rightarrow p_{\underline{u}}(Z')$. By Lemma 2.13, $p_{\underline{u}}$ is étale at all points of Z' . Thus, the base change $p_{\underline{u}}|_{p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))}$ is still étale at all points of Z' , i.e., Z' is contained inside the open subset $U'' \subseteq p_{\underline{u}}^{-1}(p_{\underline{u}}(Z'))$. But $Z' \rightarrow p_{\underline{u}}(Z')$ is an isomorphism by Lemma 2.13, so both $Z'/p_{\underline{u}}(Z')$ and $U''/p_{\underline{u}}(Z')$ are étale and hence $Z' \subseteq U''$ is open. \square

Combining Lemma 2.13 and 2.14, we can easily derive Theorem 2.3:

Proof of Theorem 2.3. First, let us recall the work done so far: We started with the reductions to $Z = V(f)$ and z in the special fibre $Z_\sigma \neq X_\sigma$ (cf. Reduction 2.4). Via Proposition 2.6 we reduced Theorem 2.3 to the case of X admitting a closed embedding $X \hookrightarrow \mathbb{A}_{\underline{x}, S}$ s.t. Z is fibrewise dense in its Zariski-closure \bar{Z} in $\mathbb{P}_{\underline{x}, S}$. In Reduction 2.8, we further assumed the point z to be closed. Next, in Proposition 2.9, we constructed the open subset $W \subseteq \mathfrak{E}^{\times sn}$ s.t. for any $\underline{u} \in W(\mathfrak{o})$ the linear projection $p_{(u_1, \dots, u_{n-1})}$ satisfies the finiteness part (1) in Theorem 2.3.

Now let $U_2 \subseteq X$ be the étale locus of $p_{\underline{u}}$. Then U_2 resp. $U := U_1 \cap U_2$ is an open neighbourhood of z in X resp. in $p_{(u_1, \dots, u_{n-1})}^{-1}(V)$. By Lemma 2.13 and 2.14, V and U satisfy the remaining claims (2) - (4) of Theorem 2.3. \square

3. OBJECTWISE STABLE \mathbb{A}^1 -CONNECTIVITY

In this section, we derive connectivity results for homotopy presheaves (i.e. “objectwise” connectivity results). These are used in the proof of our main theorem in the following section. Moreover, we show the left-completeness of the \mathbb{A}^1 -Nisnevich-local t-structure on S^1 - and on \mathbb{P}^1 -spectra. Throughout this section, let S be an arbitrary noetherian scheme of finite dimension.

Results for S^1 -spectra. We start with objectwise connectivity results for S^1 -spectra:

Proposition 3.1. *Let $U \in \mathrm{Sm}_S$ be a scheme of dimension e . Then for $E \in \mathcal{SH}_{S^1>i+e}^s(S)$, one has*

$$\pi_i^{\mathbb{A}^1}(E)(U) = [\Sigma_{S^1}^\infty(U_+)[i], L^{\mathbb{A}^1}E] = 0,$$

where $L^{\mathbb{A}^1}$ is a fibrant replacement functor for the stable \mathbb{A}^1 -Nisnevich-local model.

Remark 3.2. Note that the connectivity theorems of [Mor05] consider the sheafification $\tilde{\pi}_i^{\mathbb{A}^1}(E)$ of the presheaf $\pi_i^{\mathbb{A}^1}(E)$ from the previous Proposition 3.1. The problem is, to get an alternative bound e for the loss of connectivity, independent from the dimension of U .

Lemma 3.3. *Let $U \in \mathrm{Sm}_S$ be a scheme of dimension e . Then for $D \in \mathcal{SH}_{S^1>i+e}^s(S)$, one has*

$$[\Sigma_{S^1}^\infty(U_+)[i], L^sD] = 0.$$

Proof. It suffices to show that $[\Sigma_{S^1}^\infty(U_+), L^sD] = 0$ for $D \in \mathcal{SH}_{S^1>e}^s$: Indeed, $[\Sigma_{S^1}^\infty(U_+)[i], L^sD] \cong [\Sigma_{S^1}^\infty(U_+), L^s(D[-i])]$ as L^s is homotopy-exact. Recall that the Nisnevich-cohomological dimension is bounded by the Krull-dimension, i.e., for any sheaf G of abelian groups on Sm_S and $n > \dim(U)$, we have $[\Sigma_{S^1}^\infty(U_+), L^sHG[n]] = H_{Nis}^n(U, G) = 0$ (see e.g. [TT90, Lem. E.6.c]).

By the left-completeness of the Nisnevich-local structure, there is a filtration

$$\begin{array}{ccccccc} 0 \simeq \mathrm{holim}_{n \rightarrow \infty} L^sD_{\geq n} & \rightarrow & \dots & \longrightarrow & L^sD_{\geq e+2} & \longrightarrow & L^sD_{\geq e+1} = L^sD \\ & & & & \downarrow & & \downarrow \\ & & \dots & & L^sH\tilde{\pi}_{e+2}(D)[e+2] & & L^sH\tilde{\pi}_{e+1}(D)[e+1] \end{array}$$

and a surjection $0 = [\Sigma_{S^1}^\infty(U_+), \mathrm{holim}_n L^sD_{\geq n}] \rightarrow \lim [\Sigma_{S^1}^\infty(U_+), L^sD_{\geq n}]$ by the Milnor-lim¹-sequence. Hence, $\lim [\Sigma_{S^1}^\infty(U_+), L^sD_{\geq n}] = 0$. For $i \geq 1$, there is a long exact sequence

$$\dots \rightarrow [\Sigma_{S^1}^\infty(U_+), L^sD_{\geq e+i+1}] \rightarrow [\Sigma_{S^1}^\infty(U_+), L^sD_{\geq e+i}] \rightarrow [\Sigma_{S^1}^\infty(U_+), L^sH\tilde{\pi}_{e+i}(D)[e+i]] \rightarrow \dots$$

where the abelian group on the right-hand side is zero by the result on the Nisnevich-cohomological dimension mentioned above. For this reason, the projection $\lim [\Sigma_{S^1}^\infty(U_+), L^sD_{\geq n}] \rightarrow [\Sigma_{S^1}^\infty(U_+), L^sD_{\geq e+1}]$ is surjective and therefore we get $[\Sigma_{S^1}^\infty(U_+), L^sD_{\geq e+1}] = 0$ as desired. \square

Corollary 3.4. *Let $U \in \mathrm{Sm}_S$ be an S -pointed scheme of dimension e . For $D \in \mathcal{SH}_{S^1>i+e}^s(S)$, one has*

$$[\Sigma_{S^1}^\infty(U)[i], L^sD] = 0.$$

Proof. The basepoint $s: S \rightarrow U$ is a splitting of the structure morphism $p: U \rightarrow S$. In particular, $\dim(U) \geq \dim(S)$. Consider the distinguished triangle

$$\Sigma_{S^1}^\infty(S_+) \rightarrow \Sigma_{S^1}^\infty(U_+) \rightarrow \Sigma_{S^1}^\infty(U) \rightarrow \Sigma_{S^1}^\infty(S_+)[1].$$

If $\dim(U) > \dim(S)$ then the assertion follows from the previous Lemma 3.3 applied to the entries $\Sigma_{S^1}^\infty(U_+)$ and $\Sigma_{S^1}^\infty(S_+)[1]$ of the triangle. Now we consider the case $\dim(U) = \dim(S)$. Because of the splitting $s: S \rightarrow U$, p is surjective. Since p is smooth of relative dimension zero, it follows that p is étale. Thus, the section s itself is étale. As it is also a closed immersion, s is a component of U , i.e., $U \cong (U')_+$ for some $U' \in \mathrm{Sm}_S$ with $\dim(U') \leq \dim(U)$. The result then follows from the previous Lemma 3.3 applied to $\Sigma_{S^1}^\infty(U'_+)$. \square

Proof of Proposition 3.1. We work with the explicit model L^∞ of Theorem 1.1 as an \mathbb{A}^1 -Nisnevich-local fibrant replacement functor $L^{\mathbb{A}^1}$. By homotopy-exactness of L^∞ , we have to show that

$$[\Sigma_{S^1}^\infty(U_+), \mathrm{hocolim}_{k \rightarrow \infty} L^k(E)] = 0$$

for $U \in \mathrm{Sm}_S$ of dimension e and $E \in \mathcal{SH}_{S^1>e}^s$. Since $\Sigma_{S^1}^\infty(U_+)$ is compact, every homotopy class in question is represented by some $\Sigma_{S^1}^\infty(U_+) \rightarrow L^k(E)$. Hence, it suffices to show, that for every $k \geq 0$

$$[\Sigma_{S^1}^\infty(U_+), L^kE] = 0.$$

We argue by induction on $k \geq 0$ for all $U \in \mathrm{Sm}_S$ of dimension e and all spectra $E \in \mathcal{SH}_{S^1}^{s_{>e}}$ at once. For $k = 0$ the statement follows directly from Lemma 3.3. Let $k \geq 1$. The distinguished triangle in Remark 1.2 induces the long exact sequence

$$\cdots \rightarrow [\Sigma_{S^1}^\infty(U_+), L^{(k-1)}E] \rightarrow [\Sigma_{S^1}^\infty(U_+), L^k E] \rightarrow [\Sigma_{S^1}^\infty(U_+ \wedge \mathbb{A}^1), L^{(k-1)}E[1]] \rightarrow \cdots.$$

The abelian group on the left-hand side vanishes by the induction hypothesis on k . In order to see the vanishing of the right-hand side, we observe that

$$U \sqcup \mathbb{A}^1 \cong U_+ \vee \mathbb{A}^1 \rightarrow (U \times \mathbb{A}^1) \sqcup \mathbb{A}^1 \cong U_+ \times \mathbb{A}^1 \rightarrow U_+ \wedge \mathbb{A}^1$$

and therefore $U_+ \rightarrow (U \times \mathbb{A}^1)_+ \rightarrow U_+ \wedge \mathbb{A}^1$ is a homotopy cofibre sequence in $\mathrm{sPre}_+(S)$. This yields a distinguished triangle after applying the left-Quillen functor $\Sigma_{S^1}^\infty$. Employing this resulting distinguished triangle to $[-, L^{(k-1)}E[1]]$, we end up showing that both abelian groups $[\Sigma_{S^1}^\infty(U_+)[1], L^{(k-1)}E[1]]$ and $[\Sigma_{S^1}^\infty(U \times \mathbb{A}^1)_+, L^{(k-1)}E[1]]$ vanish, respectively. For the first group, this follows from the inductive hypothesis on k and likewise for the second, since the dimension of $U \times \mathbb{A}^1$ is $e+1$ and $E[1] \in \mathcal{SH}_{S^1}^{s_{>e+1}}$. \square

Corollary 3.5. *Let $U \in \mathrm{Sm}_S$ be an S -pointed scheme of dimension e . For $E \in \mathcal{SH}_{S^1}^{s_{>i+e}}(S)$, one has*

$$[\Sigma_{S^1}^\infty(U)[i], L^{\mathbb{A}^1}E] = 0.$$

Proof. The proof is literally the same as the proof of Corollary 3.4 using Proposition 3.1 instead of Lemma 3.3. \square

Corollary 3.6. *The \mathbb{A}^1 -Nisnevich-local t -structure on $\mathcal{SH}_{S^1}^{\mathbb{A}^1}(S)$ is left-complete and hence non-degenerate. In particular,*

$$L^{\mathbb{A}^1} \mathrm{holim}_{n \rightarrow \infty} (E_{\leq n}) \simeq \mathrm{holim}_{n \rightarrow \infty} L^{\mathbb{A}^1}(E_{\leq n}).$$

Proof. First note, that the truncation functors of the \mathbb{A}^1 -Nisnevich-local t -structure are (after inclusion to $\mathcal{SH}_{S^1}^s$) given by $L^{\mathbb{A}^1}((-)_{\leq n})$ (see Theorem 1.10). Consider a spectrum $E \in \mathrm{Spt}_{S^1}(S)$. To see that $L^{\mathbb{A}^1}E \rightarrow \mathrm{holim}_n L^{\mathbb{A}^1}(E_{\leq n})$ is an isomorphism in $\mathcal{SH}_{S^1}^s$, we may equivalently show $\mathrm{holim}_n L^{\mathbb{A}^1}(E_{\geq n}) \simeq 0$ which is implied by the triviality of the group $\pi_i(\mathrm{holim}_n L^{\mathbb{A}^1}(E_{\geq n}))(U)$ for every integer i and every $U \in \mathrm{Sm}_S$. Equivalently, we show that $\pi_i \mathrm{holim}_n (L^{\mathbb{A}^1}(E_{\geq n})(U))$ is trivial. Proposition 3.1 yields $[\Sigma_{S^1}^\infty U_+[i], L^{\mathbb{A}^1}(E_{\geq n})] = 0$ for all integers $n > i + \dim(U)$. Hence, we obtain $\lim_n \pi_i(L^{\mathbb{A}^1}(E_{\geq n})(U)) = 0$. Using Milnor's \lim^1 -sequence, it follows that the group $\pi_i \mathrm{holim}_n (L^{\mathbb{A}^1}(E_{\geq n})(U))$ is trivial: Indeed, the \lim^1 -term is trivial as the occurring groups are eventually zero. \square

Results for \mathbb{P}^1 -spectra. In this section, we show some analogous statements to those of the preceding section for \mathbb{P}^1 -spectra. These results are not needed for the rest of the paper but are of independent interest.

Proposition 3.7. *Let $U \in \mathrm{Sm}_S$ be a scheme of dimension e . Then for $E \in \mathcal{SH}_{>i+e}(S)$, one has*

$$[\Sigma_{\mathbb{P}^1}^\infty(U_+)[i]\langle q \rangle, E]_{\mathcal{SH}} = 0$$

for all $q \in \mathbb{Z}$.

Proof. Set $\mathcal{F} := [\Sigma_{\mathbb{P}^1}^\infty(U_+)[i]\langle q \rangle, -]_{\mathcal{SH}}$ for abbreviation. By the construction in Proposition 1.10, the class $\mathcal{SH}_{>i+e}$ is generated under extensions, (small) sums and cones from $\mathcal{S}[i+e+1]$. If E is obtained from an extension $E' \rightarrow E \rightarrow E''$ and \mathcal{F} vanishes on E' and E'' , then it also vanishes on E . If E is a (small) sum of objects E'_α on which \mathcal{F} vanishes, we use the homotopy-compactness of $\Sigma_{\mathbb{P}^1}^\infty(U_+)[i]\langle q \rangle$ to conclude that $\mathcal{F}(E) = 0$. Suppose that E sits in a distinguished triangle $E' \rightarrow E'' \rightarrow E \rightarrow E'[1]$ and we know the vanishing of \mathcal{F} on E'' and $E'[1]$, then we know it on E . Summing up, it suffices to show that $\mathcal{F}(\mathcal{S}[n]) = 0$ for all $n \geq i+e+1$, i.e., $[\Sigma_{\mathbb{P}^1}^\infty(U_+)[i]\langle q \rangle, \Sigma_{\mathbb{P}^1}^\infty(V_+)[n]\langle q' \rangle]_{\mathcal{SH}} = 0$ for all $V \in \mathrm{Sm}_S$ and $q' \in \mathbb{Z}$. We compute

$$\begin{aligned} [\Sigma_{\mathbb{P}^1}^\infty(U_+)[i]\langle q \rangle, \Sigma_{\mathbb{P}^1}^\infty(V_+)[n]\langle q' \rangle]_{\mathcal{SH}} &\cong [\Sigma_{\mathbb{P}^1}^\infty(U_+)[i-n], \Sigma_{\mathbb{P}^1}^\infty(V_+)\langle q' - q \rangle]_{\mathcal{SH}} \\ &\cong [\Sigma_{S^1}^\infty(U_+)[i-n], \Omega_{\mathbb{G}_m}^\infty(\Sigma_{\mathbb{G}_m}^\infty \Sigma_{S^1}^\infty(V_+)\langle q' - q \rangle)] \\ &\cong [\Sigma_{S^1}^\infty(U_+)[i-n], \mathrm{colim}_k \Omega_{\mathbb{G}_m}^k L^{\mathbb{A}^1}(\Sigma_{S^1}^\infty(V_+) \wedge \mathbb{G}_m^{\wedge(k+q'-q)})] \\ &\cong \mathrm{colim}_k [\Sigma_{S^1}^\infty(U_+)[i-n] \wedge \mathbb{G}_m^{\wedge k}, L^{\mathbb{A}^1}(\Sigma_{S^1}^\infty(V_+) \wedge \mathbb{G}_m^{\wedge(k+q'-q)})]. \end{aligned}$$

Now we use that \mathbb{A}^1 -Nisnevich-locally there is an equivalence $\mathbb{G}_m \sim \mathbb{P}^1[-1]$. Hence, it suffices to show that for all but finitely many $k \geq 0$ (and in particular, we may assume $k+q'-q \geq 0$), one has

$$[\Sigma_{S^1}^\infty(U_+ \wedge (\mathbb{P}^1)^{\wedge k})[i-n-k], L^{\mathbb{A}^1} \Sigma_{S^1}^\infty(V_+ \wedge \mathbb{G}_m^{\wedge(k+q'-q)})] = 0.$$

By the same arguments as in the proof of Proposition 3.1, this is implied by the vanishing of the group $[\Sigma_{S^1}^\infty(U \times \mathbb{P}^k)_+[i - n - k], L^{\mathbb{A}^1}\Sigma_{S^1}^\infty(V_+ \wedge \mathbb{G}_m^{\wedge(k+q'-q)})]$. Since the spectrum $\Sigma_{S^1}^\infty(V_+ \wedge \mathbb{G}_m^{\wedge(k+q'-q)})$ is in $\mathcal{SH}_{S^1 \geq 0}^s$, the result follows from Proposition 3.1 as the scheme $U \times \mathbb{P}^k$ has dimension $e + k$. \square

Corollary 3.8. *Let S be noetherian scheme of finite Krull-dimension. Then the homotopy t -structure on the motivic homotopy category $\mathcal{SH}(S)$ is left-complete and hence non-degenerate.*

Proof. Let $E \in \mathcal{SH}$. We have to show that the canonical morphism $E \rightarrow \text{holim}_{n \rightarrow \infty} E_{\leq n}$ is an isomorphism in \mathcal{SH} . Equivalently, we may show that $\text{holim}_{n \rightarrow \infty} E_{\geq n} \simeq 0$. By [Hov99, Theorem 7.3.1], this is implied by the vanishing of the homotopy classes

$$[\Sigma_{\mathbb{P}^1}^\infty(U_+)[i]\langle q \rangle, \text{holim}_{n \rightarrow \infty} E_{\geq n}]_{\mathcal{SH}}$$

in \mathcal{SH} for all $U \in \text{Sm}_S$ and all $i, q \in \mathbb{Z}$. Using Milnor's \lim^1 -sequence as in Corollary 3.6, this, in turn, is implied by the following statement: For all $U \in \text{Sm}_S$ and $i, q \in \mathbb{Z}$ there exists an integer n_0 with $[\Sigma_{\mathbb{P}^1}^\infty(U_+)[i]\langle q \rangle, E_{\geq n}] = 0$ for all $n \geq n_0$. Setting $n_0 := i + \dim(U)$, this precisely is the preceding Proposition 3.7. \square

4. STALKWISE STABLE \mathbb{A}^1 -CONNECTIVITY

In this section, we derive our main connectivity result for homotopy sheaves (a “stalkwise” connectivity result). We formulate the *shifted stable \mathbb{A}^1 -connectivity property* on the base scheme and show that this property holds for every Dedekind scheme with infinite residue fields.

Shifted stable \mathbb{A}^1 -connectivity. Let us formulate the following weaker version of Morel's stable \mathbb{A}^1 -connectivity property (cf. [Mor05, Def. 1]):

Definition 4.1. A noetherian scheme S of dimension d has the *shifted stable \mathbb{A}^1 -connectivity property*, if for every integer i and every spectrum E in $\mathcal{SH}_{S^1 \geq i}^s(S)$, the \mathbb{A}^1 -localization $L^{\mathbb{A}^1}E$ is contained in $\mathcal{SH}_{S^1 \geq i-d}^s(S)$.

Remark 4.2. For zero-dimensional noetherian schemes, this property of Definition 4.1 is the *stable \mathbb{A}^1 -connectivity property* from [Mor05, Def. 1].

Reduction 4.3. In fact, the shifted stable \mathbb{A}^1 -connectivity property is equivalent to $E \in \mathcal{SH}_{S^1 > d}^s$ implies $\tilde{\pi}_0^{\mathbb{A}^1}(E) = 0$: Let i be an integer and $E \in \mathcal{SH}_{S^1 \geq i}^s$. Then $E[d + k - i] \in \mathcal{SH}_{S^1 > d}^s$ for all integers $k \geq 1$. By assumption, we have $\tilde{\pi}_0^{\mathbb{A}^1}(E[d + k - i]) = \tilde{\pi}_{i-d-k}^{\mathbb{A}^1}(E) = 0$ for all $k \geq 1$, so $L^{\mathbb{A}^1}E \in \mathcal{SH}_{S^1 \geq i-d}^s$.

Remark 4.4. If S has the shifted stable \mathbb{A}^1 -connectivity property, then we have $\mathcal{SH}_{S^1 \geq 0}^{\mathbb{A}^1} \subseteq \mathcal{SH}_{S^1 h \geq -d}^{\mathbb{A}^1}$ by Remark 1.20.

Theorem 4.5 (Morel, [Mor05, Theorem 6.1.8]). *If S is the spectrum of a field, then S has the (shifted) stable \mathbb{A}^1 -connectivity property.*

Corollary 4.6. *If S is the spectrum of a field, then $\mathcal{SH}_{\geq 0}^{\mathbb{A}^1}(S) = \mathcal{SH}_{h \geq 0}^{\mathbb{A}^1}(S)$.*

Remark 4.7. In [Ayo06], Ayoub gave examples of base schemes that do not have the stable \mathbb{A}^1 -connectivity property: Let S/k be a connected normal surface over k an algebraically closed field, regular away from one closed singular point s . Let $S' \rightarrow S$ be a resolution with exceptional divisor E and let E_{red} be the underlying reduced subscheme. Then by loc.cit. Corollary 3.3, S does not have the stable \mathbb{A}^1 -connectivity property if $\text{Pic}_{E_{\text{red}}}$ is not \mathbb{A}^1 -invariant. Here, $\text{Pic}_{E_{\text{red}}}$ is the Nisnevich-sheafification of the presheaf $U \mapsto \text{Pic}(U \times_s E_{\text{red}})$ on $\text{Sm}_{k(s)}$. A family of concrete examples for such a surface S (due to Barbieri-Viale) is given in the example in Section 3 of loc.cit. as hypersurfaces of \mathbb{P}_k^3 . Even worse, it follows from loc.cit. Lemma 1.3 that no \mathbb{P}_k^n for $n \geq 3$ has the stable \mathbb{A}^1 -connectivity property.

In view of the previous Remark 4.7, it is natural to ask the following question.

Question 4.8. *Let S be a regular noetherian scheme of dimension d . Does S have at least the shifted stable \mathbb{A}^1 -connectivity property?*

Morel's connectivity theorem (see Theorem 4.5 above) provides a positive answer in the case of S the spectrum of a field. In the case of S a Dedekind scheme with all residue fields infinite, we get a positive answer by Corollary 4.15 below. Unfortunately, we do not have a positive or negative answer for more general base schemes. At least, a positive answer would follow from \mathbb{A}^1 -invariance of \mathbb{A}^1 -homotopy sheaves $\tilde{\pi}_k^{\mathbb{A}^1}(E)$ (see Proposition 4.13 below).

Towards stable \mathbb{A}^1 -connectivity. The singular functor $Sing: \mathbf{sPre}(S) \rightarrow \mathbf{sPre}(S)$ is given on a section $U \in \mathbf{Sm}_S$ by the diagonal of the bisimplicial set $Sing(F)(U) = F_\bullet(\Delta^\bullet \times U)$ where Δ^\bullet denotes the standard cosimplicial object in \mathbf{Sm}_S (see [MV99, Ch. 2.3.2] in the analogous situation for simplicial sheaves). An infinite alternating composition of a Nisnevich-local fibrant replacement functor L^s and $Sing$ yields an \mathbb{A}^1 -Nisnevich-local fibrant replacement functor $L^{\mathbb{A}^1}$ in the unstable setting. We refer to [MV99, Ch. 2.3.2] for details of this well-known construction.

The following Proposition 4.9 is due to Morel in the case of S being the spectrum of a field [Mor05, Lem. 6.1.4]. Note that for Proposition 4.9 and its Corollary 4.10, we work in the unstable setting.

Proposition 4.9. *Let $V \in \mathbf{Sm}_S$ be an irreducible scheme and $W \hookrightarrow V$ a non-empty open subscheme. Let $Z = (V \setminus W)_{\text{red}}$ be the reduced complement. Suppose moreover, that each point v of V admits a Nisnevich-neighbourhood V' (with pullback W' and Z' to V') together with an étale map $p: V' \rightarrow \mathbb{A}_Y^1$ in \mathbf{Sm}_S with $Z' \rightarrow Y$ finite such that*

$$\begin{array}{ccc} W' & \longrightarrow & V' \\ \downarrow & & \downarrow p \\ \mathbb{A}_Y^1 \setminus p(Z') & \longrightarrow & \mathbb{A}_Y^1 \end{array}$$

is a Nisnevich-distinguished square. Then $\tilde{\pi}_0(Sing(a_{\text{Nis}}(V/W)))$ is trivial for a_{Nis} the Nisnevich-sheafification.

Proof. We follow the proof of [Mor05, Lem. 6.1.4]. Since a simplicial presheaf F has the same zero-simplices as the simplicial presheaf $Sing(F)$, there is an epimorphism $\pi_0(F) \twoheadrightarrow \pi_0(Sing(F))$ of presheaves. As Nisnevich-sheafification preserves epimorphisms we get a natural epimorphism $\tilde{\pi}_0(F) \twoheadrightarrow \tilde{\pi}_0(Sing(F))$ of sheaves. Applying this to the discrete simplicial presheaf $F = a_{\text{Nis}}(V/W)$ and pre-composing with the epimorphism $V = a_{\text{Nis}}V \twoheadrightarrow a_{\text{Nis}}(V/W)$ of sheaves, we get a natural epimorphism

$$V \twoheadrightarrow a_{\text{Nis}}(V/W) = \tilde{\pi}_0(a_{\text{Nis}}(V/W)) \twoheadrightarrow \tilde{\pi}_0(Sing(a_{\text{Nis}}(V/W)))$$

of sheaves. Hence, it suffices to show that for each point $v \in V$ there exists a Nisnevich-neighbourhood V' such that $V' \rightarrow \tilde{\pi}_0(Sing(a_{\text{Nis}}(V'/W')))$ is zero where $W' := W \times_V V'$.

Let $v \in V$ be a point and choose the Nisnevich-neighbourhood V' from the assumption of the proposition. In particular, $V'/W' \rightarrow \mathbb{A}_Y^1/(\mathbb{A}_Y^1 \setminus Z')$ is a Nisnevich-local weak equivalence, so we may assume $V' = \mathbb{A}_Y^1$ with closed $Z' \hookrightarrow \mathbb{A}_Y^1$ and $Z' \rightarrow Y$ finite. By finiteness, the morphism $Z' \rightarrow \mathbb{A}_Y^1 \hookrightarrow \mathbb{P}_Y^1$ is proper and hence a closed immersion. Therefore we get a diagram

$$\begin{array}{ccccc} \mathbb{A}_Y^1 \setminus Z' & \longrightarrow & \mathbb{A}_Y^1 & \xrightarrow{q} & a_{\text{Nis}}\left(\mathbb{A}_Y^1 / \mathbb{A}_Y^1 \setminus Z'\right) \\ \downarrow & & \downarrow j & & \downarrow \sim \\ \mathbb{P}_Y^1 \setminus Z' & \longrightarrow & \mathbb{P}_Y^1 & \xrightarrow{q'} & a_{\text{Nis}}\left(\mathbb{P}_Y^1 / \mathbb{P}_Y^1 \setminus Z'\right) \end{array}$$

where the right vertical morphism is a Nisnevich-local weak equivalence as the left-hand square is a Zariski- and therefore a Nisnevich-distinguished square.

There exists an elementary \mathbb{A}^1 -homotopy $Y \times \mathbb{A}^1 \rightarrow \mathbb{P}_Y^1$ from the zero-section $s_0: Y \rightarrow \mathbb{A}_Y^1 \hookrightarrow \mathbb{P}_Y^1$ to the section $s_\infty: Y \rightarrow \mathbb{A}_Y^1 \hookrightarrow \mathbb{P}_Y^1$ at infinity and the latter factorizes over $\mathbb{P}_Y^1 \setminus Z'$. As by [MV99, Lem. 2.3.6] the functor $Sing$ turns elementary \mathbb{A}^1 -homotopies into objectwise homotopies, the maps $Sing(s_0)$ and $Sing(s_\infty)$ are identified in the objectwise homotopy category. The morphism $Sing(Y) \xrightarrow{\sim} Sing(\mathbb{A}_Y^1)$ is an objectwise weak equivalence by [MV99, Cor. 2.3.5]. Hence, the composition $Sing(q' \circ j)$ is the constant map to the point in the objectwise homotopy category. It follows that the same is true for $Sing(q)$, as desired. Note that the cited arguments of [MV99] are valid for the objectwise structure on simplicial presheaves. \square

Using the epimorphism $\tilde{\pi}_0(Sing(a(V/W))) \twoheadrightarrow \tilde{\pi}_0^{\mathbb{A}^1}(V/W)$ of sheaves [MV99, Cor. 2.3.22], we get:

Corollary 4.10. *In the situation of the previous Proposition 4.9, we have*

$$\tilde{\pi}_0^{\mathbb{A}^1}(V/W) = 0.$$

Remark 4.11. As explained in [Mor05, Rem. 6.1.5], $\tilde{\pi}_0^{\mathbb{A}^1}(V/W) = 0$ is not true for a general S , an irreducible $V \in \text{Sm}_S$ and $W \hookrightarrow V$ a non-empty open subscheme. For example, let S be the spectrum of a discrete valuation ring with closed point $i: \sigma \hookrightarrow S$ and open complement $j: \eta \hookrightarrow S$. Set $V := S$, $W := \eta$ and consider the \mathbb{A}^1 -Nisnevich-local homotopy cofibre sequence $j_{\#}j^*(V/W) \rightarrow V/W \rightarrow i_*L^{\mathbb{A}^1}i^*(V/W)$ from [MV99, Theorem 3.2.21]. We have $j_{\#}j^*(S/\eta) \simeq *$ and therefore $L^{\mathbb{A}^1}(V/W) \simeq i_*L^{\mathbb{A}^1}(i^*(V/W))$. On the other hand, $i^*(S/\eta) \simeq i^*(S)/i^*(\eta) \simeq \sigma/\emptyset \simeq S_{\sigma}^0$ and hence $i_*L^{\mathbb{A}^1}(i^*(V/W))$ has non-trivial $\tilde{\pi}_0$.

The following lemma is a crucial ingredient for the proof of all connectivity statements in this paper. For the special case of S the spectrum of a field, this is an observation of Morel in [Mor04, Lem. 3.3.6].

Lemma 4.12. *Let S be a noetherian scheme of dimension d with a point $s \in S$ and let $E \in \mathcal{SH}_{S^1 > d}^s(S)$ be a spectrum. Then, for any $V \in \text{Sm}_S$ and any section $f \in \pi_0^{\mathbb{A}^1}(E)(V)$ there exists an open subscheme $W \hookrightarrow V$ with $f|_W = 0$ and $s^*(W) \neq \emptyset$.*

Proof. Let $\eta_Z \in V$ be a generic point of an irreducible component Z of $s^*(V)$. The ring \mathcal{O}_{V, η_Z} has dimension at most d and

$$D := \text{Spec}(\mathcal{O}_{V, \eta_Z}) \cong \lim_{d_i: W \hookrightarrow V} W,$$

where the limit on the right-hand side is indexed by the diagram constituted by the open immersions $d_i: W \hookrightarrow V$ with W affine and $W \cap Z \neq \emptyset$. We have

$$\begin{aligned} \text{colim}_{d_i: W \hookrightarrow V} \pi_0^{\mathbb{A}^1}(E)(W) &\cong \text{colim}_{d_i: W \hookrightarrow V} [W_+, L^{\mathbb{A}^1}E] \\ &\cong \text{colim}_{d_i: W \hookrightarrow V} [p_{\#}(W_+), L^{\mathbb{A}^1}E] \\ &\cong \text{colim}_{d_i: W \hookrightarrow V} [W_+, p^*(L^{\mathbb{A}^1}E)] \\ &\cong \text{colim}_{d_i: W \hookrightarrow V} [W_+, L^{\mathbb{A}^1}(p^*E)] && \text{(by Lemma 1.4)} \\ &\cong \text{colim}_{d_i: W \hookrightarrow V} [d_{i\#}d_i^*(S_V^0), L^{\mathbb{A}^1}(p^*E)] \\ &\cong \text{colim}_{d_i: W \hookrightarrow V} [d_i^*(S_V^0), d_i^*L^{\mathbb{A}^1}(p^*E)] \\ &\cong [d^*(S_V^0), d^*L^{\mathbb{A}^1}(p^*E)] && \text{(by (3) of Lemma 1.5)} \\ &\cong [S_D^0, L^{\mathbb{A}^1}(d^*p^*E)] && \text{(by (5) Lemma 1.5)} \\ &\cong \pi_0^{\mathbb{A}^1}(d^*p^*E)(D), \end{aligned}$$

where $p: V \rightarrow S$ is the structural morphism. Using the Quillen adjoint pair $(p_{\#}, p^*)$, we see that p^* preserves the connectivity. By Lemma 1.5, the same is true for d^* , so $d^*(p^*E)$ is contained in $\mathcal{SH}_{S^1 > d}^s(D)$. By Proposition 3.1, we get $\pi_0^{\mathbb{A}^1}(d^*p^*E)(D) = 0$ as D has dimension at most d .

The restrictions of $f \in \pi_0^{\mathbb{A}^1}(E)(V)$ induce an element of the set $\text{colim}_{d_i} \pi_0^{\mathbb{A}^1}(E)(W) = 0$ from the left-hand side of the chain of equations above. This means that there exists an open subscheme $W \hookrightarrow V$ with $W \cap Z \neq \emptyset$ and $f|_W = 0$. Since $Z \subseteq s^*(V)$, we have $s^*(W) \neq \emptyset$. \square

As mentioned above, a positive answer to Question 4.8 would follow from \mathbb{A}^1 -invariance of \mathbb{A}^1 -homotopy sheaves:

Proposition 4.13. *Let S be a regular noetherian scheme of dimension d . Let i be an integer and $E \in \mathcal{SH}_{\geq i}^s(S)$ such that the sheaf $\tilde{\pi}_k^{\mathbb{A}^1}(E)$ is \mathbb{A}^1 -invariant for all integers $k < i - d$. Then $L^{\mathbb{A}^1}E \in \mathcal{SH}_{S^1 \geq i-d}^s(S)$.*

Proof. First note, that for any open immersion $j: S' \hookrightarrow S$ the functor j^* preserves \mathbb{A}^1 -invariance of (simplicial) presheaves and $j^*\tilde{\pi}_0 \cong \tilde{\pi}_0j^*$. In particular, our assumptions on the spectrum are stable under restriction to open subschemes of the base. Let E be a spectrum in $\mathcal{SH}_{S^1 > d}^s$ with \mathbb{A}^1 -invariant homotopy sheaves $\tilde{\pi}_k^{\mathbb{A}^1}(E)$ in degrees $k \leq 0$. To prove Proposition 4.13, it is enough to show that $\tilde{\pi}_0^{\mathbb{A}^1}(E)$ is trivial (cf. Reduction 4.3). We argue by induction on the dimension d of the base S . The case $d = 0$ is [Mor05, Theorem 6.1.8]. Let $d > 0$. By Corollary 1.6, we may assume that S is local with closed point $i: \sigma \hookrightarrow S$. Take a connected scheme $V \in \text{Sm}_S$ with structure morphism $p: V \rightarrow S$ and a point $v \in V$. It suffices to show that the Nisnevich-stalk of $\tilde{\pi}_0^{\mathbb{A}^1}(E)$ at (V, v) is trivial. By the induction hypothesis, we may assume that v lies in the fibre over σ as $S \setminus \sigma$ has Krull-dimension strictly smaller than d . Moreover, we may assume that $i^*(V)$ is connected. Let $f_{(V, v)}$ be a germ in this stalk. We have to show that $f_{(V, v)}$ is trivial: After possibly refining (V, v) Nisnevich-locally, we may assume that $f_{(V, v)}$ is induced by a section

f in $\pi_0^{\mathbb{A}^1}(E)(V)$. By Lemma 4.12, there exists an open subscheme $W \hookrightarrow V$ with $f|_W = 0$ and $i^*(W) \neq \emptyset$. Clearly, we may assume that $v \notin W$. The cofibre sequence $W_+ \rightarrow V_+ \rightarrow V/W$ induces an exact sequence

$$0 \rightarrow \tilde{\pi}_0(L^{\mathbb{A}^1}E)(V/W) \rightarrow \tilde{\pi}_0(L^{\mathbb{A}^1}E)(V) \rightarrow \tilde{\pi}_0(L^{\mathbb{A}^1}E)(W)$$

of homotopy sheaves. Here, we write $\tilde{\pi}_0(L^{\mathbb{A}^1}E)(V/W)$ for $\text{Hom}(V/W, \tilde{\pi}_0(L^{\mathbb{A}^1}E))$. Since the restriction of f to W is trivial, it suffices to show that $\tilde{\pi}_0(L^{\mathbb{A}^1}E)(V/W)$ is trivial. By [MV99, Theorem 3.2.21] there exists an \mathbb{A}^1 -Nisnevich-local homotopy cofibre sequence

$$j_{\#}j^*(V/W) \rightarrow V/W \rightarrow i_*L^{\mathbb{A}^1}i^*(V/W)$$

inducing a long exact sequence

$$\cdots \rightarrow [i_*L^{\mathbb{A}^1}i^*(V/W), \tilde{\pi}_0^{\mathbb{A}^1}(E)] \rightarrow [V/W, \tilde{\pi}_0^{\mathbb{A}^1}(E)] \rightarrow [j_{\#}j^*(V/W), \tilde{\pi}_0^{\mathbb{A}^1}(E)]$$

by the \mathbb{A}^1 -Nisnevich-local fibrancy of $\tilde{\pi}_0^{\mathbb{A}^1}(E) = \tilde{\pi}_0(L^{\mathbb{A}^1}E)$. For the latter, note that a sheaf considered as a discrete simplicial presheaf is Nisnevich-locally fibrant. The set on the right-hand side equals $[j^*(V/W), j^*\tilde{\pi}_0^{\mathbb{A}^1}(E)]$ and $j^*\tilde{\pi}_0^{\mathbb{A}^1}(E) \cong \tilde{\pi}_0^{\mathbb{A}^1}(j^*E)$ is trivial by induction. The triviality of the set on the left-hand side follows from the triviality of $\tilde{\pi}_0(i_*L^{\mathbb{A}^1}i^*(V/W))$. By [Spi14, Prop. 4.2], the latter is zero, if $\tilde{\pi}_0^{\mathbb{A}^1}(i^*(V/W)) = 0$. Since $i^*(V)$ is irreducible and $i^*(W)$ is non-empty, we conclude by [Mor05, Lem. 6.1.4]. \square

The one dimensional case. Using the Gabber-presentation provided by Theorem 2.3, we can give a positive answer to Question 4.8 for a Dedekind scheme S with infinite residue fields (see Corollary 4.15 below). This essentially boils down to the following local henselian case:

Theorem 4.14. *Let S be the spectrum of a henselian discrete valuation ring with infinite residue field. Then S has the shifted stable \mathbb{A}^1 -connectivity property: $E \in \mathcal{SH}_{S^1 \geq i}^s(S)$ implies $L^{\mathbb{A}^1}E \in \mathcal{SH}_{S^1 \geq i-1}^s(S)$.*

Proof of Theorem 4.14. By Reduction 4.3 we have to show that the sheaf $\tilde{\pi}_0^{\mathbb{A}^1}(E)$ is trivial for every $E \in \mathcal{SH}_{S^1 \geq 1}^s$. Take a connected scheme $V \in \text{Sm}_S$ with structure morphism $p: V \rightarrow S$ and a point $v \in V$. It suffices to show that the Nisnevich-stalk of $\tilde{\pi}_0^{\mathbb{A}^1}(E)$ at (V, v) is trivial. Let $f_{(V, v)}$ be a germ in this stalk. Possibly refining (V, v) Nisnevich-locally, we may assume that $f_{(V, v)}$ is induced by a section f in $\pi_0^{\mathbb{A}^1}(E)(V)$. By Lemma 4.12, there exists an open subscheme $W \hookrightarrow V$ with $f|_W = 0$ and $s^*(W) \neq \emptyset$, where $s := p(v)$. Clearly, we may assume that $v \notin W$. The homotopy cofibre sequence $W_+ \rightarrow V_+ \rightarrow V/W$ induces a long exact sequence

$$\cdots \rightarrow \pi_0(L^{\mathbb{A}^1}E)(V/W) \rightarrow \pi_0(L^{\mathbb{A}^1}E)(V) \rightarrow \pi_0(L^{\mathbb{A}^1}E)(W) \rightarrow \cdots$$

of homotopy presheaves. Since the restriction of f to W is trivial, f is the image of an element in the group $\pi_0(L^{\mathbb{A}^1}E)(V/W)$, i.e., a morphism $g: V/W \rightarrow (L^{\mathbb{A}^1}E)_0$ in the (unstable) objectwise (pointed) homotopy category. We want to show the triviality of the germ $f_{(V, v)}$, so it is enough to show that $\tilde{\pi}_0(g)$ is trivial.

As the adjunction $(\Sigma_{S^1}^{\infty}, (-)_0)$ is a Quillen-adjunction for the \mathbb{A}^1 -Nisnevich-local model, $(L^{\mathbb{A}^1}E)_0$ is \mathbb{A}^1 -Nisnevich-local. Therefore, the morphism g factors through $h: L^{\mathbb{A}^1}(V/W) \rightarrow (L^{\mathbb{A}^1}E)_0$ and it suffices to show that $\tilde{\pi}_0(h)$ is trivial. For the latter, we show the triviality of $\tilde{\pi}_0(L^{\mathbb{A}^1}(V/W))$: Since Zariski-locally we may always shrink V around v , we can assume that its fibre $s^*(V)$ is connected so $s^*(W)$ has strictly positive codimension in $s^*(V)$. By Theorem 2.1, the assumptions of Corollary 4.10 are fulfilled so $\tilde{\pi}_0^{\mathbb{A}^1}(V/W) = 0$ holds. \square

Corollary 4.15. *Let S be Dedekind scheme and assume that all of its residue fields are infinite. Then S has the shifted stable \mathbb{A}^1 -connectivity property: $E \in \mathcal{SH}_{S^1 \geq i}^s(S)$ implies $L^{\mathbb{A}^1}E \in \mathcal{SH}_{S^1 \geq i-1}^s(S)$.*

Proof. Clearly, we may assume that S is connected. Again, by Reduction 4.3 we just have to show that the sheaf $\tilde{\pi}_0^{\mathbb{A}^1}(E)$ is trivial for every $E \in \mathcal{SH}_{S^1 \geq 1}^s$. It follows from Corollary 1.6 that $\tilde{\pi}_0(L^{\mathbb{A}^1}E)$ is trivial if and only if $\tilde{\pi}_0(\mathfrak{s}^*L^{\mathbb{A}^1}E)$ is trivial for all $s \in S$. By the previous Theorem 4.14 (in the case of s a closed point) resp. by [Mor05, Theorem 6.1.8] (in the case of s a generic point) the latter holds for all s . \square

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